Harmonic Analysis
Homework 2
Bruno Poggi
Department of Mathematics, University of Minnesota

September 25, 2016

Notation
Throughout, \( \mathbb{N} := \{1, 2, 3, \ldots\} \), \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), \( B_r(x) \) is the ball of radius \( r \) with center \( x \) in the understood metric space (usually \( \mathbb{R} \)). Let \( X \) be a compact topological vector space. Then by \( C(X) \) we denote the space of continuous real-valued functions on \( X \). In particular, \( C[0,1] \) is the space of continuous (hence, uniformly continuous) real-valued functions on \([0,1]\). We always think of \( C(X) \) as a Banach space endowed with the norm

\[
\|f\|_{C(X)} := \sup_{x \in X} |f(x)|.
\]

Unless otherwise noted, by \((\cdot, \cdot)\) we denote the \( L^2 \) product of two functions on the measure space understood from context. For a definition of approximate identity, see [2] Definition 1.2.15 (although we use a different order of the axioms). The function \( 1_X : \mathbb{R} \to \{0,1\} \) is the indicator function of the set \( X \). The function \( \delta_{a,b} \) is the Kronecker delta, written as

\[
\delta_{a,b} := \begin{cases} 
1, & a = b \\
0, & a \neq b
\end{cases}
\]

1 Problem 1

Poisson kernel: for \( 0 < r < 1 \), define the Poisson kernel as

\[
P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{2\pi i n t} \quad (1.1)
\]

for \( t \in \mathbb{T} \).
(a) Prove that \( P_r(t) = \text{Re} \frac{1 + re^{2\pi it}}{1 - re^{2\pi it}} = \frac{1 - r^2}{1 - 2r \cos(2\pi t) + r^2}. \)

(b) Deduce that the family \( P_r(t) \) is an approximate identity as \( r \to 1^- \) and observe that \( P_r(t) \) is decreasing in \( t \) on \([0, 1/2)\).

(c) Define the conjugate Poisson kernel

\[
Q_r(t) = -i \sum_{n=-\infty}^{\infty} \text{sgn}(n)r^{|n|}e^{2\pi int}
\]

for \( t \in \mathbb{T} \). Show that \( Q_r(t) = \frac{2r \sin(2\pi t)}{1 - 2r \cos(2\pi t) + r^2} \).

(d) Let \( f \in L^1(\mathbb{T}) \) be real-valued. Prove that the function \( z \to (P_r * f)(t) + i(Q_r * f)(t) \) is analytic in \( z = re^{2\pi it} \) on the open unit disc \( \{ z \in \mathbb{C} : |z| < 1 \} \).

(e) Conclude that the functions \( u(z) = (P_r * f)(t) \) and \( v(z) = (Q_r * f)(t) \) are conjugate harmonic functions on the open unit disc. In which sense does \( f \) represent the boundary value of \( u \)?

(see Grafakos, Ex. 3.1.7 and 4.1.4)

**Solution.**

(a) Since \( r \in (0,1) \), the series in (1.1) converges absolutely. Since, by the formula for geometric series, we can write

\[
\sum_{n=0}^{\infty} r^n e^{2\pi int} = \frac{1}{1 - re^{2\pi it}}, \quad (1.2)
\]

\[
\sum_{n=-\infty}^{0} r^{-n} e^{2\pi int} = \frac{1}{1 - re^{-2\pi it}}, \quad (1.3)
\]

it follows

\[
P_r(t) = \frac{1}{1 - re^{2\pi it}} + \frac{1}{1 - re^{-2\pi it}} - 1, \quad (1.4)
\]

but the first two terms above are complex conjugates of one another; so that

\[
P_r(t) = -1 + 2 \text{Re} \frac{1}{1 - re^{2\pi it}} = \text{Re} \frac{1 + re^{2\pi it}}{1 - re^{2\pi it}},
\]

while also from (1.4) we obtain after summing fractions that

\[
P_r(t) = \frac{1 - r^2}{1 - r(e^{2\pi it} + e^{-2\pi it}) + r^2} = \frac{1 - r^2}{1 - 2r \cos(2\pi t) + r^2},
\]

2
as desired.

(b) Since $P_r(t)$ is even, we can write

$$
\int_T P_r(t) \, dt = \int_T \frac{1 + re^{2\pi it}}{1 - re^{2\pi it}} \, dt = 1 + \int_T \frac{2re^{2\pi it}}{1 - re^{2\pi it}} \, dt = 1 \tag{1.5}
$$

since

$$
\int_T \frac{2re^{2\pi it}}{1 - re^{2\pi it}} \, dt = \frac{1}{\pi i} \left[ \ln(1 - r) - \ln(1 - r) \right] = 0,
$$

which proves the first axiom. Since $r > 0$, we note $1 - r^2 > 0$ and

$$
1 - 2r \cos(2\pi t) + r^2 \geq 1 - 2r + r^2 = (1 - r)^2 > 0
$$

so that $P_r(t)$ is strictly positive, and so the second axiom is automatically satisfied by (1.5). Now let $\delta \in (0, 1/2)$. We note that for every $t \in [\delta, 1/2)$,

$$
1 - 2r \cos(2\pi t) + r^2 \geq 1 - 2r \left( \cos(2\pi \delta) + 1 - 1 \right) + r^2 = (1 + r)^2 - 2r(1 + \cos(2\pi \delta)),
$$

and since

$$
\lim_{r \to 1^-} \left( (1 + r)^2 - 2r(1 + \cos(2\pi \delta)) \right) = 4 - 2(1 + \cos(2\pi \delta)) > 0,
$$

we have

$$
\int_{T \setminus B_2(0)} P_r(t) \, dt = 2 \int_0^{1/2} \frac{1 - r^2}{1 - 2r \cos(2\pi t) + r^2} \, dt \leq 2 \int_0^{1/2} \frac{1 - r^2}{1 - 2r \cos(2\pi t) + r^2} \, dt
$$

$$
\leq 2 \left( 1 + r \right)^2 - 2r(1 + \cos(2\pi \delta)) \left( \frac{1}{2} - \delta \right) \to 0 \text{ as } r \to 1^-,
$$

which finishes the proof that the family $\{P_r\}$ is an approximate identity.

(c) Subtract (1.2) from (1.3) to get that

$$
iQ_r(t) = \sum_{n=-\infty}^{\infty} \text{sgn}(n)e^{2\pi int} = \frac{1}{1 - re^{2\pi it}} - \frac{1}{1 - re^{-2\pi it}} = 2 \text{ Im } \frac{1}{1 - re^{2\pi it}} \tag{1.6}
$$

while

$$
\frac{1}{1 - re^{2\pi it}} = \frac{1}{1 - re^{2\pi it}} \frac{1 - re^{-2\pi it}}{1 - re^{-2\pi it}} = \frac{1 - re^{-2\pi it}}{1 - 2r \cos(2\pi t) + r^2},
$$
which finally implies
\[ iQ_r(t) = \frac{2r \sin(2\pi t)}{1 - 2r \cos(2\pi t) + r^2}, \]
which gives the desired formula.

\textbf{(d)} Given \( z \) in the open unit disc, there exist unique \( r \in (0, 1), \ t \in [0, 1] \), for which we can write \( z := re^{2\pi it} \). Using (1.4) and (1.6), we observe
\[
(P_r * f)(t) = \int_T f(\tau) P_r(t - \tau) d\tau = \int_T f(\tau) \left[ \frac{1}{1 - ze^{-2\pi i\tau}} + \frac{1}{1 - \overline{z}e^{2\pi i\tau}} - 1 \right] d\tau,
\]
and
\[
i(Q_r * f)(t) = \int_T f(\tau) \left[ \frac{1}{1 - ze^{-2\pi i\tau}} - \frac{1}{1 - \overline{z}e^{2\pi i\tau}} \right] d\tau,
\]
so that
\[
(P_r * f)(t) + i(Q_r * f)(t) = 2 \int_T f(\tau) \frac{1}{1 - ze^{-2\pi i\tau}} d\tau - \int_T f(\tau) d\tau. \tag{1.7}
\]
The last term is a constant, so we focus on the first term on the right-hand side of (1.7). Using the geometric series and Fubini’s Theorem (which is valid since \( f \in L^1(T) \)), we note
\[
2 \int_T f(\tau) \frac{1}{1 - e^{-2\pi i\tau}} d\tau = 2 \int_T f(\tau) \sum_{n=0}^{\infty} (re^{2\pi i(t-\tau)})^n d\tau
\]
\[
= 2 \sum_{n=0}^{\infty} \frac{r^n e^{2\pi int}}{n} \int_T f(\tau) e^{-2\pi int} d\tau = 2 \sum_{n=0}^{\infty} 2 \hat{f}(n) z^n.
\]
It follows from this last observation and (1.7) that the function \( F(z) \) is a sum of analytic functions, hence itself analytic on the open unit disc, as desired.

\textbf{(e)} Since \( F(z) \) is analytic, the Cauchy-Riemann equations are satisfied and each of \( u(z), v(z) \) must be harmonic, by construction. Also, they are conjugate by construction, since \( u, v \) are the real and imaginary parts respectively of \( F(z) \), an analytic function.

Since \( \{P_r\} \) is an approximate identity, we have that
\[
\|P_r * f - f\|_1 \longrightarrow 0 \quad \text{as} \ r \rightarrow 1^-,
\]
so, if we let \( C_a \) be the circle of radius \( a \) in \( \mathbb{C} \), and we interpret \( u|_{C_1} \) to be the limit of \( u|_{C_r} \) as \( r \rightarrow 1^- \), then
\[
u|_{C_1} = f
\]
in \( L^1(T) \).
2 Problem 2

(Fejér’s lemma). Let \( f \in L^1(\mathbb{T}) \) and \( g \in L^\infty(\mathbb{T}) \). Prove that

\[
\lim_{n \to \infty} \int_\mathbb{T} f(t) g(nt) \, dt = \hat{f}(0) \hat{g}(0). \tag{2.1}
\]

**Solution.** Identify \( \mathbb{T} \) with the interval \([0,1]\). It is of course understood that the function \( g \) is extended to \( \mathbb{R} \) as a 1–periodic extension. We split the proof into three main steps: first we prove (2.1) in the case that both \( f, g \) are trigonometric polynomials; in this case we can identify \( f \) and \( g \) almost everywhere with their respective absolutely convergent Fourier series. Next, we use the previous step to prove (2.1) when \( f \) is a trigonometric polynomial and \( g \in L^\infty(\mathbb{T}) \). The final step then will give the desired result by using the density of trigonometric polynomials in \( L^1(\mathbb{T}) \).

**Step 1.** Assume both \( f, g \) are trigonometric polynomials, and say \( \{a_\ell\}, \{b_k\} \) are the lists of coefficients of \( f, g \) respectively. It is easy to see that the Fourier coefficients of \( f, g \) satisfy

\[
\hat{f}(\ell) = \begin{cases} a_\ell, & |\ell| \leq N \\ 0, & |\ell| > N \end{cases}, \quad \hat{g}(k) = \begin{cases} b_k, & |k| \leq M \\ 0, & |k| > M \end{cases},
\]

where \( N, M \) are the degrees of \( f, g \) as trigonometric polynomials respectively. As such, we can write

\[
f(t) = \sum_{\ell=-N}^{N} \hat{f}(\ell)e^{2\pi i \ell t}, \quad g(t) = \sum_{k=-M}^{M} \hat{g}(k)e^{2\pi i kt}, \quad \forall t \in \mathbb{T}.
\]
Therefore,

\[\int_0^1 f(t)g(nt) \, dt = \int_0^1 \sum_{\ell=-N}^{N} \hat{f}(\ell)e^{2\pi i\ell t} \sum_{k=-M}^{M} \hat{g}(k)e^{2\pi iknt} \, dt\]

\[= \sum_{\ell=-N}^{N} \sum_{k=-M}^{M} \hat{f}(\ell)\hat{g}(k) \int_0^1 e^{2\pi i(\ell+kn)t} \, dt\]

\[= \sum_{\ell=-N}^{N} \sum_{k=-M}^{M} \hat{f}(\ell)\hat{g}(k)\delta_{\ell,-kn} = \sum_{k=-M}^{M} \hat{f}(-kn)\hat{g}(k)\]

\[= \hat{f}(0)\hat{g}(0) + \sum_{k=-M}^{M} \hat{f}(-kn)\hat{g}(k). \quad (2.2)\]

So, for all \(n > N = 1 + \deg f\), we have

\[\hat{f}(-kn) \equiv 0, \quad \forall k \in \mathbb{Z}\setminus\{0\},\]

and so (2.2) implies

\[\int_0^1 f(t)g(nt) \, dt = \hat{f}(0)\hat{g}(0), \quad \forall n \geq N = N(f), \quad (2.3)\]

which in particular implies (2.1), in this case.

**Step 2.** Suppose \(f\) is a trigonometric polynomial as above, \(g \in L^\infty(\mathbb{T})\), and fix \(\varepsilon > 0\). Then \(g \in L^1(\mathbb{T})\), and since trigonometric polynomials are dense in \(L^1(\mathbb{T})\) (see, for instance, [3] Theorem 2.12), we can procure a trigonometric polynomial \(p(t)\) such that

\[\|g - p\|_1 < \frac{\varepsilon}{2\|f\|_\infty},\]
We note
\[
\left| \int_0^1 f(t)g(nt) \, dt - \hat{f}(0)\hat{g}(0) \right| \leq \int_0^1 |f(t)||g(nt) - p(nt)| \, dt + \int_0^1 |f(t)p(nt) \, dt - \hat{f}(0)\hat{p}(0)|
+ |\hat{f}(0)\hat{p}(0) - \hat{f}(0)\hat{g}(0)|,
\]
\[
\leq \|f\|_\infty \frac{1}{n} \int_0^n |g(t) - p(t)| \, dt + |\hat{f}(0)||\hat{p}(0) - \hat{g}(0)| + \int_0^1 |f(t)p(nt) \, dt - \hat{f}(0)\hat{p}(0)|
\]
and using the periodicity of \(g, p\), we arrive at
\[
\left| \int_0^1 f(t)g(nt) \, dt - \hat{f}(0)\hat{g}(0) \right| \leq 2\|g\|_\infty \|f - p\|_1 + \int_0^1 |f(t)p(nt) \, dt - \hat{f}(0)\hat{p}(0)|. \quad (2.4)
\]
Now choose \(N\) so large that for \(n \geq N\), the last term on the right-hand side of (2.4) is identically 0, which can be done as in (2.3). It follows that
\[
\left| \int_0^1 f(t)g(nt) \, dt - \hat{f}(0)\hat{g}(0) \right| < \varepsilon, \quad \forall n \geq N = N(f),
\]
which proves (2.1) in this case.

**Step 3.** Now say \(f \in L^1(\mathbb{T})\), \(g \in L^\infty(\mathbb{T})\), and fix \(\varepsilon > 0\). Similar to above, we procure a trigonometric polynomial \(q(t)\) such that
\[
\|f - q\|_1 < \frac{\varepsilon}{4\|g\|_L^\infty(\mathbb{T})}, \quad (2.5)
\]
and so after adding and subtracting certain terms like above, we arrive at
\[
\left| \int_0^1 f(t)g(nt) \, dt - \hat{f}(0)\hat{g}(0) \right| \leq 2\|g\|_\infty \|f - q\|_1 + \int_0^1 q(t)g(nt) - \hat{q}(0)\hat{g}(0) \right| . \quad (2.6)
\]
Finally, per the conclusion of the above step we can choose \(N\) so large that for \(n \geq N\) we have
\[
\left| \int_0^1 q(t)g(nt) - \hat{q}(0)\hat{g}(0) \right| < \frac{\varepsilon}{2},
\]
which, when combined with (2.6) and (2.5), gives (2.1) in the desired generality. □

**Remark.** The above proof actually shows that the limit in (2.1) does not depend on \( g \); that is, for each \( \varepsilon > 0 \), there exists \( N = N(f) \) such that \( n \geq N \) implies

\[
\left| \int_0^1 f(t)g(nt) - \hat{f}(0)\hat{g}(0) \right| < \varepsilon.
\]

3 Problem 3

([3] Chapter 1, Section 4, Exercise 2). Let \( f \in L^1(\mathbb{T}) \). Show that if \( \sum |\hat{f}(n)||n|^m < \infty \), then \( f \) is \( m \) times continuously differentiable.

Deduce that, if \( \hat{f}(n) = O(|n|^{-k}) \) for \( k > 2 \) and

\[
m = \begin{cases} 
  k - 2 & \text{if } k \text{ is integer}, \\
  \lfloor k \rfloor - 1 & \text{otherwise},
\end{cases}
\]

then \( f \) is \( m \) times continuously differentiable.

**Solution.** Fix \( m \in \mathbb{N} \). Since

\[
\sum_{n \in \mathbb{Z}} |\hat{f}(n)| \leq |\hat{f}(0)| + \sum_{n \in \mathbb{Z}} |\hat{f}(n)||n|^m < \infty
\]

it follows that \( f \) equals its Fourier series almost everywhere by the Fourier Inversion Theorem (see, for instance, [2] Proposition 3.2.5), so from here on we identify \( f \) with its Fourier series. Then, for each \( k = 1, \ldots, m \), since

\[
\sum_{n \in \mathbb{Z}} |\hat{f}(n)||n|^k \leq \sum_{n \in \mathbb{Z}} |\hat{f}(n)||n|^m < \infty
\]

it follows that the series

\[
\sum_{n \in \mathbb{Z}} (2\pi in)^k \hat{f}(n)e^{2\pi int}
\]

is absolutely convergent uniformly for all \( t \in [0,1] \). Hence \( f \) is \( m \)-times differentiable with

\[
f^{(m)}(t) = \sum_{n \in \mathbb{Z}} (2\pi in)^m \hat{f}(n)e^{2\pi int}.
\]

Now let \( \hat{f}(n) = O(|n|^{-k}) \) with \( k > 2 \) and \( m = \lfloor k - 2 \rfloor \). We note that

\[
\sum_{n \in \mathbb{Z}} |\hat{f}(n)||n|^m \leq C \sum_{n \in \mathbb{Z}\backslash\{0\}} |n|^{-k}|n|^m \leq C \sum_{n \in \mathbb{Z}\backslash\{0\}} |n|^{-(k-\lfloor k - 2 \rfloor)} < \infty
\]
since 

\[ k - \lceil k - 2 \rceil > 1. \]

Consequently, from our previous argument, we see that \( f \) is \( m \) times continuously differentiable. \( \square \)

4 Problem 4

([3], Chapter 1, Section 5, Exercise 5) Let \( f \in L^1(\mathbb{T}) \) and \( \hat{f}(n) = O(|n|^{-k}) \). Show that \( f \) is \( m \)-times differentiable with \( f^{(m)} \in L^2 \) provided \( k > m + \frac{1}{2} \).

Solution. Fix \( m \in \mathbb{N} \) and say \( k > m + \frac{1}{2} \). Since

\[ \lceil k - 2 \rceil \geq \lceil m + \frac{1}{2} - 2 \rceil = m - 1 \]

it follows by the result of the previous problem that \( f \) is \( m - 1 \) times continuously differentiable, and moreover we can write

\[ f^{(m-1)}(t) = \sum_{n \in \mathbb{Z}} (2\pi in)^{m-1} \hat{f}(n) e^{2\pi int}. \]

By assumption, there exists \( \eta > 0 \) such that \( k = m + \frac{1}{2} + \eta \). Let

\[ a_n := (2\pi in)^m \hat{f}(n), \quad n \in \mathbb{Z} \]

and note that

\[ \sum_{n \in \mathbb{Z}} |a_n|^2 \leq (2\pi)^{2m} \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^2 n^{2m} \leq (2\pi)^{2m} C \sum_{n \in \mathbb{Z}} |n|^{2m-2m-1-\eta} < \infty \]

since \( \eta > 0 \). Hence \( \{a_n\} \in l^2 \). Thus ([3], Theorem 5.5) there exists a unique \( g \in L^2(\mathbb{T}) \) such that \( \hat{g}(n) = a_n \). We now intend to show \( g \) is the derivative of \( f^{(m-1)} \) in the a.e. sense. First note \( g \in L^1(\mathbb{T}) \). Again from Theorem 5.5 in [3], we observe that for every \( x \in \mathbb{T} \),

\[ \int_{\mathbb{T}} g(t) \mathbf{1}_{[0,x]}(t) dt = \sum_{n \in \mathbb{Z}} \hat{g}(n) \mathbf{1}_{[0,x]}(n) = \sum_{n \in \mathbb{Z}} (2\pi in)^{m-1} \hat{f}(n) \left[ e^{2\pi in} - 1 \right] = f^{(m-1)}(x) - f^{(m-1)}(0), \]

whence the Fundamental Theorem of Calculus (see, for instance, [1] Theorem 3.36) gives that \( f^{(m-1)} \) is differentiable a.e. on \([0,1] \), absolutely continuous on \([0,1] \), and \( f^{(m)} = g \) a.e.. This ends the proof. \( \square \)
5 Problem 5

([3] Chapter 1, Section 3, Exercise 1). Let $f \in L^1(\mathbb{T})$ and let $0 < \alpha \leq 1$. Assume that $f$ satisfies Lipschitz (Hölder) condition of order $\alpha$ at the point $t_0$. Prove that

$$|\sigma_N f(t_0) - f(t_0)| \leq CN^{-\alpha} \quad \text{for } \alpha < 1,$$

and

$$|\sigma_N f(t_0) - f(t_0)| \leq C \frac{\log N}{N} \quad \text{for } \alpha = 1,$$

where $\sigma_N f$ is the $N^{th}$ Fejér (Cesàro) mean of the Fourier series of $f$.

**Solution.** Identify the torus $\mathbb{T}$ with the interval $[-1/2, 1/2]$. By assumption, there exists $K \geq 0$ such that

$$|f(t_0 + h) - f(t_0)| \leq Kh^\alpha, \quad \forall |h| \leq 1/2.$$

(5.3)

Now, we calculate:

$$|\sigma_N f(t_0) - f(t_0)| \leq \int_{\mathbb{T}} F_N(h)|f(t_0 - h) - f(t_0)| \, dh \leq 2K \int_0^{1/2} F_N(h)h^\alpha \, dh,$$

(5.4)

where in the second inequality we used (5.3) and the evenness of $F_N$. At this point, it is possible to use estimate (3.10) from [3] and split the integral up at the point $1/N$ to get the desired inequalities. We proceed in a different way, which sharpens somewhat the constants in (5.1) and (5.2). We can write

$$\int_0^{1/2} F_N(h)h^\alpha \, dh = \int_0^{1/2} \frac{1}{N + 1} \frac{\sin^2((N + 1)\pi h)}{\sin^2(\pi h)} h^\alpha \, dh$$

$$= \frac{1}{N + 1} \sum_{k=0}^{\frac{k+1}{2(N+1)}} \int_{\frac{k}{2(N+1)}}^{\frac{k+1}{2(N+1)}} \frac{\sin^2((N + 1)\pi h)}{\sin^2(\pi h)} h^\alpha \, dh$$

$$\leq \frac{1}{N + 1} \left[ \sum_{k=1}^{N} \frac{1}{\sin^2 \left( \frac{k}{2(N+1)} \pi \right)} \int_{\frac{k}{2(N+1)}}^{\frac{k+1}{2(N+1)}} \sin^2((N + 1)\pi h)h^\alpha \, dh + \int_0^{\frac{1}{2(N+1)}} \frac{\sin^2((N + 1)\pi h)}{\sin^2(\pi h)} h^\alpha \, dh \right]$$

10
where in the last inequality we used the fact that \( \sin^2(x) \) is an increasing function on \([0, \pi/2]\). The last term is seen to be of order \( O((N+1)^{-\alpha}) \), as follows:

\[
\frac{1}{N+1} \int_0^{\pi/(N+1)} \frac{\sin^2((N+1)/\pi)h^\alpha}{\sin^2(\pi h)} dh \\
\leq \frac{1}{N+1} \int_0^{\pi/(N+1)} \frac{\sin^2((N+1)/\pi)h}{((N+1)/\pi)^2} \frac{\pi^2 h}{2} \frac{(N+1)^2}{\pi^2 h^2} h^\alpha dh \\
\leq (N+1) \frac{\pi^2}{4} \int_0^{\pi/(N+1)} h^\alpha dh \leq (N+1) \frac{\pi^2}{4(1+\alpha)} \left( \frac{1}{2(N+1)} \right)^{1+\alpha} \\
\leq \frac{\pi^2}{2^{3+\alpha}(1+\alpha)} (N+1)^{-\alpha},
\]

(5.5)

where in the second inequality we used the fact that

\[
1 \leq \frac{x}{\sin x} \leq \frac{\pi}{2}, \quad \forall x \in [0, \pi/2],
\]

(5.6)

and we remark that the second inequality in (5.6) can be obtained by virtue of \( \frac{\pi}{\sin x} \) being increasing on \([0, \pi/2]\). Moreover, we observe

\[
\frac{1}{N+1} \sum_{k=1}^{N} \frac{1}{\sin^2 \left( \frac{k}{2(N+1)} \pi \right)} \frac{1}{\sin^2 \left( \frac{k}{2(N+1)} \right)} \int_{\frac{k}{2(N+1)}}^{\frac{k+1}{2(N+1)}} \frac{\sin^2((N+1)/\pi)h^\alpha}{\sin^2(\pi h)} dh \\
= \frac{1}{N+1} \sum_{k=1}^{N} \frac{1}{\sin^2 \left( \frac{k}{2(N+1)} \pi \right)} \left( \frac{1}{(N+1)\pi} \right)^{1+\alpha} \int_{\frac{k}{2(N+1)}}^{\frac{k+1}{2(N+1)}} \sin^2(h) h^\alpha dh \\
\leq (N+1)^{-\alpha} \frac{4}{\pi^{3+\alpha}} \sum_{k=1}^{N} \frac{\pi^2}{\sin^2 \left( \frac{k}{2(N+1)} \pi \right)} \frac{(k+1)^2}{k^2} \int_{\frac{k}{2}}^{\frac{k+1}{2}} \sin^2(h) \left( \frac{k+1}{\pi} \right)^\alpha dh \\
\leq (N+1)^{-\alpha} \frac{1}{2^{2+\alpha}} \sum_{k=1}^{N} \frac{(k+1)^\alpha}{k^2} \leq (N+1)^{-\alpha} \frac{1}{4} \sum_{k=1}^{N} \frac{1}{k^{1+(1-\alpha)}},
\]

(5.7)
where in the first equality we did a change of variables $h \mapsto (N + 1)\pi h$, and in the second inequality we used (5.6) again. If $\alpha < 1$, then the series

\[ S(\alpha) := \sum_{k=1}^{\infty} \frac{1}{k^{1+(1-\alpha)}} \]

is convergent, and hence, combining (5.5), (5.7) and (5.4), we have proved

\[ |\sigma_N f(t_0) - f(t_0)| \leq 2K \left[ \frac{1}{4} S(\alpha) + \frac{\pi^2}{2^{3+\alpha}(1+\alpha)} \right] (N + 1)^{-\alpha}, \quad \alpha < 1, \]

which gives (5.1). If instead $\alpha = 1$, in this case we note that (recall this calculation from our solution to Homework 1)

\[ \sum_{k=1}^{N} \frac{1}{k} \leq \log N, \]

so that from (5.5), (5.7) and (5.4), we have

\[ |\sigma_N f(t_0) - f(t_0)| \leq 2K \left[ \frac{1}{4} \log N + \frac{\pi^2}{32} \right] (N + 1)^{-1}, \quad \alpha = 1, \]

which proves (5.2). \qed

**Remark.** In the above method, the constants can be sharpened even further.

**References**

