1 Problem 1

(Exercise 2.1.6 in [1]) Let \( M_S(f)(x) \) be the supremum of the average of \(|f|\) over all rectangles containing \( x \) with sides parallel to the axes. The operator \( M_s \) is called the strong maximal function.

(a) Prove that \( M_S \) maps \( L^p \) to itself for \( p > 1 \).

(b) Prove that \( M_S \) is not weak type \((1,1)\).

Solution.

Proof of (a). Let \( f \in L^p(\mathbb{R}^n) \) and \( h_1, \ldots, h_n \) be positive real numbers. If \( n = 1 \), the strong maximal function is exactly the Hardy-Littlewood maximal function and there is nothing to show. So suppose \( n \geq 2 \). Since \( f \in L^p(\mathbb{R}^n), p > 1 \), then \( f \) is locally integrable on \( \mathbb{R}^n \). By Fubini’s Theorem, we can write

\[
M_S(f)(x) = \sup_{h_1, \ldots, h_n} \frac{1}{h_1 h_2 \cdots h_n} \int_{-h_n}^{h_n} \cdots \int_{-h_2}^{h_2} \int_{-h_1}^{h_1} |f(x-y)| dy_1 dy_2 \cdots dy_n. \tag{1.1}
\]

where \( y = (y_1, y_2, \ldots, y_n), x = (x_1, x_2, \ldots, x_n) \). Let \( R \) be the rectangle of integration in (1.1). From (1.1) we can see

\[
M_S(f)(x) \leq \sup_{h_n} \frac{1}{h_n} \int_{-h_n}^{h_n} \sup_{h_2} \frac{1}{h_2} \int_{-h_2}^{h_2} \sup_{h_1} \frac{1}{h_1} \int_{-h_1}^{h_1} |f(x-y)| dy_1 dy_2 \cdots dy_n. \tag{1.2}
\]
Since the Hardy-Littlewood maximal function $M$ maps $L^p$ to itself (that is, it is a bounded linear operator on $L^p$ to itself), then the map

$$f(x) \mapsto \sup_{h_1} \frac{1}{h_1} \int_{-h_1}^{h_1} |f(x-y)| dy,$$

maps $L^p(\mathbb{R})$ to itself (in the above map, only the first coordinate is taken as a variable; the rest are fixed. This can be done in an adequate a.e. sense in $\mathbb{R}$, per Fubini’s Theorem). It is clear then that the right-hand side of (1.2) is simply the composition of $n$ bounded linear operators on $L^p(\mathbb{R})$, hence itself a bounded linear operator on $L^p(\mathbb{R}^n)$. Thus $M_S(f)$ maps $L^p(\mathbb{R}^n)$ to itself, for $p > 1$.

**Proof of (b).** If we get a counterexample in the case $n = 2$, then we have a counterexample for any number of dimensions, by taking a constant extension to the rest of the variables. So assume the number of dimensions is 2, and let $1_S$ be the characteristic function of the square $S = [0, 1] \times [0, 1]$. Write $g(x) := M_S(1_S)$. By a straightforward computation, on the region

$$D := \{ (x, y) \mid x \geq 1, y \geq 1 \}$$

it is clear that

$$g(x) = g(x_1, x_2) = \frac{1}{x_1 x_2}.$$

Since

$$\alpha d_g(\alpha) \geq \alpha |D \cap \{|g| > \alpha\}|,$$

it will suffice to show that the right-hand side of the above inequality is unbounded for $\alpha > 0$. By construction, it is clear that $\alpha > 1 \implies d_g(\alpha) = 0$, and so without loss of generality we can restrict ourselves to $\alpha < 1$. We calculate that

$$|D \cap \{|g| > \alpha\}| = \int \int \frac{1}{x_1 x_2} 1 dx = \int_1^1 \frac{1}{x_1} dx_1 dx_2 = \frac{1}{\alpha} \ln \left( \frac{1}{\alpha} \right) - \left( \frac{1}{\alpha} - 1 \right), \quad \forall \alpha \in (0, 1),$$

(1.3)

so that

$$\alpha |D \cap \{|g| > \alpha\}| = \ln \left( \frac{1}{\alpha} \right) - 1 + \alpha$$

and this last expression goes to $+\infty$ as $\alpha \searrow 0$. It then follows that

$$\sup_{\alpha > 0} \alpha d_g(\alpha) = +\infty$$

so that $g \notin L^{p, \infty}$. This shows that $M_S$ does not map $L^p$ into weak $L^p$. \qed
2 Problem 2

(Exercise 1.1.7 in [1]) Let $f_1, \ldots, f_N \in L^{p,\infty}(X, \mu)$ for $p \in [1, \infty)$. Prove that
\[
\| \sum_{j=1}^{N} f_j \|_{p,\infty} \leq N \sum_{j=1}^{N} \| f_j \|_{p,\infty}.
\] (2.1)

Solution.

Observe that
\[
\gamma \left( \sum_{j=1}^{N} d_{f_j} \left( \frac{\gamma}{N} \right) \right)^{\frac{1}{p}} \leq \gamma \left( \sum_{j=1}^{N} d_{f_j} \left( \frac{\gamma}{N} \right) \right)^{\frac{1}{p}} = N \sum_{j=1}^{N} \gamma d_{f_j} \left( \frac{\gamma}{N} \right)^{\frac{1}{p}}
\]
\[
\leq N \sum_{j=1}^{N} \| f_j \|_{p,\infty}, \quad \forall \gamma > 0,
\] (2.2)
where the first inequality occurs due to the summation property of the distribution function, the last inequality occurs by definition of the weak $L^p$ quasi-norm, and the second inequality occurs since $p \geq 1$ and the fact that for any sequence $\{a_j\}$ of non-negative reals we have
\[
\left( \sum_{j=1}^{N} a_j \right)^{\theta} \leq \sum_{j=1}^{N} a_j^{\theta},
\]
and this last inequality, in turn is observed to be a consequence of Minkowski’s Inequality for the space $\ell^p$. Taking supremum over all possible $\gamma > 0$ in (2.2) yields the desired result. \qed

3 Problem 3

(Exercise 1.1.12 in [1]) (Normability of weak $-L^p$ for $p > 1$) Let $(X, \mu)$ be a $\sigma$–finite measure space and let $p > 1$. Define
\[
\| f \|_{p,\infty} = \sup_{0 < \mu(E) < \infty} \mu(E)^{\frac{1}{p} - 1} \int_{E} |f| d\mu.
\] (3.1)

(a) Prove that
\[
\| f \|_{p,\infty} \leq \frac{p}{p-1} \| f \|_{p,\infty},
\] (3.2)
where $\| f \|_{p,\infty}$ is the usual quasi-norm on $L^{p,\infty}$. 

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(b) Prove that
\[ \|f\|_{p,\infty} \leq \|f\|_{p,\infty}. \] (3.3)

(c) Prove that \( \|f\|_{p,\infty} \) is a norm and deduce that \( L^{p,\infty} \) is norm-able for \( p > 1 \).

**Solution.**

**Proof of (a).** Let \( E \) be a measurable set in \( X \) with \( \mu(E) < \infty \). We can then consider the restriction of \( \mu \) to \( E \), and so the measure space \( (E, \mu) \) is finite (in particular, \( \sigma \)-finite). It follows we can write
\[ \int_E |f| \, d\mu = \int_0^\infty d_f \{ x \in E \mid |f| > \alpha \} \, d\alpha \leq \int_0^\infty \min \left( \mu(E), \frac{1}{\alpha^p} \|f\|_{p,\infty}^p \right) \, d\alpha \leq \int_0^\infty \min \left( \mu(E), \frac{1}{\alpha^{p-1}} \|f\|_{p,\infty} \right) \, d\alpha \]
let \( A := \mu(E)^{-\frac{1}{p}} \|f\|_{p,\infty} \) then
\[ \int_E |f| \, d\mu \leq \int_0^A \mu(E) \, d\alpha + \int_A^\infty \frac{1}{\alpha^p} \|f\|_{p,\infty} \, d\alpha = \mu(E)^{1-\frac{1}{p}} \|f\|_{p,\infty} + \|f\|_{p,\infty} \frac{1}{1-p} \alpha^{1-p} \big|_A^\infty \]
\[ = \frac{p}{p-1} \|f\|_{p,\infty} \mu(E)^{1-\frac{1}{p}} \]
\[ \implies \mu(E)^{\frac{1}{p}-1} \int_E |f| \, d\mu \leq \frac{p}{p-1} \|f\|_{p,\infty}, \quad \forall E \text{ measurable s.t. } \mu(E) < \infty, \]
from which (3.2) follows immediately.

**Proof of (b).** If \( \|f\|_{p,\infty} = \infty \) there is nothing to show, so suppose for the sake of contradiction that it is finite. For any \( \alpha > 0 \), let \( E = \{|f| > \alpha\} \). Thus we note \( \mu(E) = d_f(\alpha) \), and we observe that
\[ \mu(E)^{\frac{1}{p}-1} \int_E |f| \, d\mu \geq \mu(E)^{\frac{1}{p}-1} \left( \alpha \mu(E) \right) = \alpha d_f(\alpha)^{\frac{1}{p}}. \] (3.4)
Now, \( X \) is a \( \sigma \)-finite measure space, so there exist nested measurable \( X_k \subset X, k = 1, 2, \ldots \) with \( \mu(X_k) < \infty \) for each \( k \) and such that \( X = \bigcup_{k=1}^\infty X_k \). It is easy to show, by
the Monotone Convergence Theorem, that
\[ \int_{X_k \cap E} |f| \, d\mu \nearrow \int_E |f| \, d\mu, \quad \text{as } k \nearrow \infty \]
and
\[ \mu(X_k \cap E) \nearrow \mu(E), \quad \text{as } k \nearrow \infty, \]
whence
\[
\mu(E)^{\frac{1}{p} - 1} \int_E |f| \, d\mu = \lim_{k \to \infty} \mu(X_k \cap E)^{\frac{1}{p} - 1} \int_{X_k \cap E} |f| \, d\mu \\
\leq \sup_k \mu(X_k \cap E)^{\frac{1}{p} - 1} \int_{X_k \cap E} |f| \, d\mu \leq \|f\|_{p, \infty}.
\]
This last inequality, in combination with (3.4), imply that
\[ \alpha d_f(\alpha)^{\frac{1}{p}} \leq \|f\|_{p, \infty}, \quad \forall \alpha > 0, \]
and the desired result follows upon taking supremum over \( \alpha > 0 \).

**Proof of (c).** Due to (3.2) and (3.3), it is clear that \( f = 0 \) is the only function (up to \( \mu - a.e. \)) with \( \|f\|_{p, \infty} = 0 \). If \( c \in \mathbb{C} \) is a scalar, then from (3.1) it is obvious that
\[ \|c\|_{p, \infty} = \|f\|_{p, \infty}, \]
since constants "come out" of the integral operation. Further, if \( f, g \in L^{p, \infty} \), then by (3.3), \( \|f\|_{p, \infty} < \infty \) and \( \|g\|_{p, \infty} < \infty \), and since
\[
\int_E |f + g| \, d\mu \leq \int_E |f| \, d\mu + \int_E |g| \, d\mu
\]
the Triangle inequality
\[ \|f + g\|_{p, \infty} \leq \|f\|_{p, \infty} + \|g\|_{p, \infty} \]
follows from the definition (3.1) easily. Thus \( \|f\|_{p, \infty} \) is a norm on \( L^{p, \infty} \), \( p > 1 \), so that this space is normable \( \square \)
4 Problem 4

(Exercise 2.1.13 in [1]) Observe that in the proof of the weak-(1, 1) boundedness of the Hardy-Littlewood maximal function we have actually obtained the inequality

$$\lambda \cdot \mu\left( \{ M(f) > \lambda \} \right)^{\frac{1}{p}} \leq 3^n \mu\left( \{ M(f) > \lambda \} \right)^{\frac{1}{p} - 1} \int_{\{ M(f) > \lambda \}} |f| d\mu$$  \hspace{1cm} (4.1)

for \( \lambda > 0 \) and \( f \) locally integrable. Use this fact and the previous exercise to deduce that \( M \) maps \( L^{p,\infty} \) into itself.

Solution.

Suppose \( f \in L^{p,\infty} \). By the result of Problem 3 it follows \( f \) is locally integrable. By way of contradiction, suppose first that the set \( \{ M(f) > \lambda \} \) has infinite Lebesgue measure. Write \( \mathbb{R}^n = \bigcup_{N=1}^\infty B_N(0) \), and it is easy to show by the Monotone Convergence Theorem that

$$\int_{B_N(0) \cap \{ M(f) > \lambda \}} |f| d\mu \nearrow \int_{\{ M(f) > \lambda \}} |f| d\mu, \quad \text{as } N \nearrow \infty$$

and

$$\mu\left( B_N(0) \cap \{ M(f) > \lambda \} \right) \nearrow \mu\left( \{ M(f) > \lambda \} \right), \quad \text{as } N \nearrow \infty.$$  

So from (4.1) we observe that

$$\infty = \lambda \cdot \mu\left( \{ M(f) > \lambda \} \right)^{\frac{1}{p}} \leq 3^n \lim_{N \to \infty} \mu\left( B_N(0) \cap \{ M(f) > \lambda \} \right)^{\frac{1}{p} - 1} \int_{B_N(0) \cap \{ M(f) > \lambda \}} |f| d\mu$$

$$\leq 3^n \sup_N \mu\left( B_N(0) \cap \{ M(f) > \lambda \} \right)^{\frac{1}{p} - 1} \int_{B_N(0) \cap \{ M(f) > \lambda \}} |f| d\mu$$

$$\leq 3^n \|f\|_{p,\infty} < \infty$$

which is a contradiction. It follows that if \( f \) is in weak \( L^p \) then the Lebesgue measure of \( \{ M(f) > \lambda \} \) is finite for any \( \lambda > 0 \). Hence the right-hand side of (4.1) is bounded above by \( 3^n \|f\|_{p,\infty} \). Taking supremum over all \( \lambda > 0 \) in (4.1) shows that

$$\| M(f) \|_{p,\infty} \leq 3^n \|f\|_{p,\infty},$$

as desired. \( \square \)
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References