Introduction. This document provides the sketch of a solution to problem 53 from section 12.3 of the book *Calculus, Early Transcendentals*, 8th Edition, by James Stewart. We will go over this problem in class on Tuesday, September 13th.

12.3 53. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$  

Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

Solution. Let $L$ be the line given by

$$ax + by + c = 0,$$

and let $(p, q)$ be the coordinates of a point on the line $L$. Then for any point $(x, y)$ on the line $L$, by subtracting the equation of the line for the points $(x, y)$ and $(p, q)$ we get

$$a(x - p) + b(y - q) = 0,$$

which can be written as

$$(a, b) \cdot (x - p, y - q) = 0.$$  

So, any point on the line $L$ is given by the above equation. It follows that the vector $a := (a, b)$ is orthogonal to any vector parallel to the line $L$. Now consider the vector

$$x := (x_1 - p, y_1 - q),$$

which can be thought of as representing the line connecting the points $(x_1, y_1)$ and $(p, q)$. Then the distance $d$ from the point $P_1(x_1, y_1)$ to the line $L$ is given by the magnitude of
the scalar projection of \( \mathbf{x} \) onto \( \mathbf{a} \), because the latter is the length of the side of a right triangle with two of the vertices being \( P_1 \) and \( (p, q) \), and the other vertex lying on \( L \). In other words,

\[
d = |\text{comp}_a \mathbf{x}| = \left| \frac{x \cdot a}{|a|} \right| = \left| \frac{(x_1 - p, y_1 - q) \cdot (a, b)}{|(a, b)|} \right| = \frac{|ax_1 - ap + by_1 - bq|}{\sqrt{a^2 + b^2}},
\]

and using the fact that the point \( (p, q) \) lies on \( L \), we get

\[
\implies d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}},
\]

as desired (the calculation asked at the end is really easy; you should do it as an exercise). \( \square \)