MATH 8802: Functional Analysis
Homework 2
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Notation
Unless explicitly stated otherwise, \( \mathbb{N} = \{1, 2, 3, \ldots\} \), and \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \). Throughout, \( B(x, r) \) is the open ball of radius \( r \) with center \( x \) in the understood metric space (usually \( \mathbb{R} \) or \( \mathbb{C} \)). When we say a function is smooth, we mean that it is infinitely differentiable on its domain of definition. \( Du = \nabla u \) is the gradient of \( u \). For \( X, Y \) normed spaces and \( A : X \to Y \) a linear operator, we use the notation \( \mathcal{D}(A) \) for the domain of \( A \), \( \mathcal{R}(A) \) for the range of \( A \). \( \partial_j u \) := \( \frac{\partial u}{\partial x_j} \).

1 Problem 1
Let \( A \) be a closed operator on a Banach space \( X \) and \( B \in L(X) \). Assume \( \rho(A) \neq \emptyset \). Prove that \( A \) commutes with \( B \) (that is, \( B \) maps \( \mathcal{D}(A) \) into \( \mathcal{D}(A) \)) if and only if \( BR(\lambda, A) = R(\lambda, A)B \) for some \( \lambda \in \rho(A) \). Also prove that if the latter is true, then \( BR(\lambda, A) = R(\lambda, A)B \) for all \( \lambda \in \rho(A) \).

Solution. We first show that if \( A \) commutes with \( B \) (in the above sense), then for each \( \lambda \in \rho(A) \), one has \( BR(\lambda, A) = R(\lambda, A)B \). Fix \( \lambda \in \rho(A) \). Then \( (\lambda - A) : \mathcal{D}(A) \to X \) is an invertible operator, \( \mathcal{R}(\lambda - A) = X \), and the operator \( R(\lambda, A) = (\lambda - A)^{-1} \) is bounded on \( X \). Fix \( x \in X \). Then \( R(\lambda, A)x \in \mathcal{D}(A) \). Per the hypothesis that \( B \) maps \( \mathcal{D}(A) \) into \( \mathcal{D}(A) \), we have in particular that \( BR(\lambda, A)x \in \mathcal{D}(A) \). Observe that

\[
(\lambda - A)R(\lambda, A)Bx = Bx = B(\lambda - A)R(\lambda, A)x = B\lambda R(\lambda, A)x - BAR(\lambda, A)x
\]

\[
= \lambda BR(\lambda, A)x - ABR(\lambda, A)x = (\lambda - A)BR(\lambda, A)x,
\]

where in the fourth equality we used the fact that \( AB = BA \) on \( \mathcal{D}(A) \). Thus, we have

\[
(\lambda - A)\left[R(\lambda, A)Bx - BR(\lambda, A)x\right] = 0.
\]
Since $\lambda - A$ is injective, it follows that $R(\lambda, A)Bx = BR(\lambda, A)x$. Since $x \in X$ was arbitrary, we conclude that $BR(\lambda, A) = R(\lambda, A)B$, for each $\lambda \in \rho(A)$. Since $\rho(A) \neq \emptyset$, the desired conclusion follows.

Now we show that if there exists $\lambda \in \rho(A)$ such that $BR(\lambda, A) = R(\lambda, A)B$, then $A$ commutes with $B$ on $\mathcal{D}(A)$. So fix $\lambda \in \rho(A)$ such that $BR(\lambda, A) = R(\lambda, A)B$. Observe that this identity implies that

$$\mathcal{R}(BR(\lambda, A)) = \mathcal{R}(R(\lambda, A)B) \subset \mathcal{R}(R(\lambda, A)) = \mathcal{D}(A),$$

whence the surjectivity of $R(\lambda, A)$ onto $\mathcal{D}(A)$ implies that $B$ maps $\mathcal{D}(A)$ into $\mathcal{D}(A)$. Hence the operator $AB$ is well-defined. Let $x \in \mathcal{D}(A)$. Then there exists $y \in X$ such that $x = R(\lambda, A)y$, or equivalently, $(\lambda - A)x = y$. Notice that

$$ABx = ABR(\lambda, A)y = (-\lambda + A)R(\lambda, A)By + \lambda R(\lambda, A)By = -By + \lambda BR(\lambda, A)y$$

$$= -B\left[y - \lambda R(\lambda, A)y\right] = -B\left[(\lambda - A)x - \lambda x\right] = BAx,$$

which gives the desired result.

Finally, suppose that $BR(\lambda, A) = R(\lambda, A)B$ holds for some $\lambda \in \rho(A)$. By the above argument, it follows that $B$ maps $\mathcal{D}(A)$ into $\mathcal{D}(A)$, and $AB = BA$ on $\mathcal{D}(A)$. In turn, as was shown further above, this implies that $BR(\lambda, A) = R(\lambda, A)B$ for all $\lambda \in \rho(A)$. □

2 Problem 2

Let $X = \ell^2$. Find all three parts of $\sigma(A)$ for the operator $A \in L(X)$ given by

$$Ax = (0, \varepsilon_1x_1, \varepsilon_2x_2, \ldots) \quad (x = (x_1, x_2, \ldots) \in X),$$

where $\varepsilon_j > 0, j = 1, 2, \ldots$ and $\varepsilon_j \to 0$ as $j \to \infty$.

Solution. First, let $\lambda \in \mathbb{C}$ be arbitrary. Define $\varepsilon_0 = 0$, and for each $x = (x_j)_{j=1}^\infty \in \ell^2$, write $x_0 = 0$. It is clear that $(\varepsilon_jx_j) \in \ell^2$, since $x_j \in \ell^\infty$. Observe that

$$(\lambda - A)x = \lambda x - Ax = (\lambda x_j) - (\varepsilon_j - 1)x_{j-1} = (\lambda x_j - \varepsilon_{j-1}x_{j-1}), \quad \text{for each } x \in \ell^2.$$

It is straightforward that $(\lambda - A)x \in \ell^2$, and if we write $y = (y_j)_{j=1}^\infty = (\lambda - A)x$, then the above computation shows that

$$y_j = \lambda x_j - \varepsilon_{j-1}x_{j-1}, \quad \text{for } j \in \mathbb{N}.$$  \hspace{1cm} (2.1)

We claim that $0 \in \sigma_r(A)$. That is, $\lambda = 0$ belongs to the residual spectrum of $A$. To this end, we show that the operator $(\lambda - A) \equiv -A$ is injective into $\ell^2$, but $\mathcal{R}(-A)$ is not dense in $\ell^2$. Let $x, x' \in \ell^2$ and suppose that $-Ax = -Ax'$. By (2.1), we have that

$$-\varepsilon_{j-1}x_{j-1} = -\varepsilon_{j-1}x'_{j-1}, \quad \text{for each } j \in \mathbb{N},$$
and hence
\[ x_j = x'_j, \quad \text{for each } j \in \mathbb{N}, \]
so that \( x = x' \) as elements in \( \ell^2 \). Thus \(-A\) is injective. On the other hand, by the definition of \( A \) it is straightforward that
\[ \mathcal{R}(-A) \subset \left\{ y = (y_j)_{j=1}^\infty \in \ell^2 : y_1 = 0 \right\} =: S. \]
But \( S \) is a closed proper subspace of \( \ell^2 \), and so in particular \( \mathcal{R}(-A) \) cannot be dense in \( \ell^2 \). For the sake of being explicit, define the sequence \( x = (x_j)_{j=1}^\infty \) given by \( x_1 = 1, x_j = 0 \) for each \( j \geq 2 \). Then \( x \in \ell^2 \), and for any element \( y \in S \),
\[ \| x - y \|_{\ell^2} \geq 1, \]
as expected. Hence \( 0 \in \sigma_r(A) \).

We now claim that if \( \lambda \neq 0 \), then \( \lambda \in \rho(A) \). We first show that \((\lambda - A)\) is surjective onto \( \ell^2 \). Let \( y \in \ell^2 \). Define the sequence \( x = (x_j)_{j=1}^\infty \) by
\[ x_j = \frac{1}{\lambda} \left[ y_j + \varepsilon_{j-1} x_{j-1} \right], \quad \text{for } j \in \mathbb{N}, \tag{2.2} \]
where \( x_0 = \varepsilon_0 = 0 \). We need to show that \( x \in \ell^2 \). Since \( \varepsilon_j \to 0 \) as \( j \to \infty \), there exists \( J_\lambda \in \mathbb{N}, J_\lambda \geq 3 \), such that for each \( j \geq J_\lambda - 1 \), we have
\[ \varepsilon_j \leq \frac{|\lambda|}{2}. \]

Now fix \( J \in \mathbb{N} \) such that \( J \geq J_\lambda \). Consider the following computation:
\[
\left( \sum_{j=1}^{J} |x_j|^2 \right)^{\frac{1}{2}} = \frac{1}{|\lambda|} \left( \sum_{j=1}^{J} \left| y_j + \varepsilon_{j-1} x_{j-1} \right|^2 \right)^{\frac{1}{2}} \leq \frac{1}{|\lambda|} \left( \sum_{j=1}^{J} |y_j|^2 \right)^{\frac{1}{2}} + \frac{1}{|\lambda|} \left( \sum_{j=1}^{J} |\varepsilon_{j-1} x_{j-1}|^2 \right)^{\frac{1}{2}}
\leq \frac{1}{|\lambda|} \| y \|_{\ell^2} + \frac{1}{|\lambda|} \left( \sum_{j=1}^{J_\lambda-1} |\varepsilon_{j-1} x_{j-1}|^2 \right)^{\frac{1}{2}} + \frac{1}{|\lambda|} \left( \sum_{j=J_\lambda}^{J} |\varepsilon_{j-1} x_{j-1}|^2 \right)^{\frac{1}{2}}
\leq \frac{1}{|\lambda|} \| y \|_{\ell^2} + \frac{1}{|\lambda|} \left( \sum_{j=2}^{J_\lambda-1} |\varepsilon_{j-1} x_{j-1}|^2 \right)^{\frac{1}{2}} + \frac{1}{2} \left( \sum_{j=J_\lambda-1}^{J-1} |x_j|^2 \right)^{\frac{1}{2}}. \tag{2.3} \]
The last term on the right-hand side of (2.3) gets absorbed by the left-hand side, to obtain the estimate
\[
\left( \sum_{j=1}^{J} |x_j|^2 \right)^{\frac{1}{2}} \leq \frac{2}{|\lambda|} \| y \|_{\ell^2} + \frac{2}{|\lambda|} \left( \sum_{j=2}^{J_\lambda-1} |\varepsilon_{j-1} x_{j-1}|^2 \right)^{\frac{1}{2}}, \quad \text{for each } J \geq J_\lambda. \tag{2.4} \]
Note that the right-hand side of (2.4) is a constant with respect to \( J \). It follows that 
\[
\sum_{j=1}^{\infty} |x_j|^2 < \infty,
\]
so that \( x \in \ell^2 \). Therefore, \( x \in D(A) = \ell^2 \), and by its definition (2.2) and the computation (2.1) we have that \((\lambda - A)x = y\). Since \( y \) was arbitrary in \( \ell^2 \), we have that \((\lambda - A)\) is surjective onto \( \ell^2 \). In particular, \( A(\lambda - A) \) is dense in \( \ell^2 \).

Actually, \((\lambda - A)\) is also injective into \( \ell^2 \): Given any \( x' \in \ell^2 \) which satisfies \((\lambda - A)x' = (\lambda - A)x\) for \( x \in \ell^2 \) defined by (2.2), then \( x' \) must satisfy the system (2.1) for \( y = (\lambda - A)x \). Since \( \lambda \neq 0 \), this is equivalent to \( x' \) satisfying the relations (2.2), so that \( x' = x \) in \( \ell^2 \).

Therefore, the operator \((\lambda - A)\) is invertible on \( X \). It remains to show that \((\lambda - A)^{-1}\) is a bounded operator on \( X \). Given \( y \in \ell^2 \), \( x = (\lambda - A)^{-1}y \in \ell^2 \) is given by (2.2), and therefore (2.4) holds as before. Observe that \( \max_{j \in \mathbb{N}} \varepsilon_j \) exists since \( \varepsilon_j \to 0 \) as \( j \to \infty \).

Let 
\[
M := \max \left\{ |\lambda|, \max_{j \in \mathbb{N}} \varepsilon_j \right\}.
\]

We claim that
\[
|x_j| \leq \frac{M^{j-1}}{|\lambda|^j} \sum_{k=1}^{j} |y_k|,
\]
for each \( j \in \mathbb{N} \). (2.5)

We proceed to prove the claim by induction. By (2.2) (recall we set \( x_0 = \varepsilon_0 = 0 \)), note that
\[
|x_1| = \frac{1}{|\lambda|} |y_1|,
\]
which shows that the base case holds. Suppose (2.5) holds for some \( j \in \mathbb{N} \). By (2.2) we have
\[
|x_{j+1}| \leq \frac{1}{|\lambda|} |y_{j+1}| + \frac{M}{|\lambda|} |x_j| \leq \left( \frac{M}{|\lambda|} \right)^{j+1} |y_{j+1}| + \frac{M^{j-1}}{|\lambda|^j} \sum_{k=1}^{j} |y_k| = \frac{M^{j+1}}{|\lambda|^{j+1}} \sum_{k=1}^{j+1} |y_k|,
\]
which is (2.5) at the level \( j+1 \). Since \( j \) was arbitrary, it follows that (2.5) holds for each \( j \in \mathbb{N} \). By the Cauchy-Schwarz inequality, (2.5) implies the estimate
\[
|x_j| \leq \frac{M^{j-1}}{|\lambda|^j} \sqrt{j} \|y\|_{\ell^2},
\]
for each \( j \in \mathbb{N} \). (2.6)

Hence,
\[
\frac{2}{|\lambda|} \left( \sum_{j=2}^{J} \varepsilon_{j-1} |x_{j-1}|^2 \right)^{1/2} \leq \frac{2M}{|\lambda|} \sum_{j=1}^{J-2} |x_j| \leq \left[ 2 \sum_{j=1}^{J-2} \frac{M^j}{|\lambda|^{j+1} \sqrt{j}} \right] \|y\|_{\ell^2}. \tag{2.7}
\]

Finally, using (2.7) on (2.4) and letting \( J \to \infty \), we arrive at the estimate
\[
\|(\lambda - A)^{-1}y\|_{\ell^2} \leq C \|y\|_{\ell^2},
\]
where $C$ is a constant depending only on $\lambda$ and $(\varepsilon_j)$. Thus $(\lambda - A)^{-1}$ is bounded on $\ell^2$, which completes the proof that $\lambda \in \rho(A)$.

Since $\rho(A) = \mathbb{C}\setminus \{0\}$ and $0 \in \sigma_r(A)$, it follows that $\sigma(A) = \sigma_r(A)$, $\sigma_p(A) = \sigma_c(A) = \emptyset$. □

Method 2: Approach through spectral radius. (Note to grader: please grade method 1.) For any $n \in \mathbb{N}$ and $x \in \ell^2$, it is easy to see by induction that

$$(A^n x)_j = 0, \ j = 1, \ldots, n, \quad (A^n x)_j = x_j \prod_{k=j-n}^{j-1} \varepsilon_k, \ j \geq n + 1. \quad (2.8)$$

whence we estimate that

$$\|A^n x\|^2 = \sum_{j=n+1}^{\infty} |(A^n x)_j|^2 = \sum_{j=n+1}^{\infty} \left( \prod_{k=j-n}^{j-1} \varepsilon_k \right)^2 |x_j|^2 \leq \sup_{j \geq n+1} \left( \prod_{k=j-n}^{j-1} \varepsilon_k \right)^2 \|x\|^2,$$

(the supremum on the right-hand side exists since $\varepsilon_j \to 0$ as $j \to \infty$), so that

$$\|A^n\|_{\mathcal{B}(\phi_0)} \leq \sup_{j \geq n+1} \left( \prod_{k=j-n}^{j-1} \varepsilon_k \right)^{\frac{1}{n}} \leq \sup_{j \geq n+1} \frac{1}{n} \sum_{k=j-n}^{j-1} \varepsilon_k, \quad (2.9)$$

where we used the Arithmetic Mean-Geometric Mean Inequality. The right-hand side drops to 0 as $n \to \infty$, because $\varepsilon_j \to 0$ as $j \to \infty$ (probably not going to write down a proof of this here, but the idea is this: there are uniformly finitely many terms in the sequence $\{\varepsilon_j\}$ which are large; as $n \to \infty$, we are taking average over so many elements of $\{\varepsilon_j\}$ that the large terms of the sequence are neglected in the limit). Therefore, $(2.9)$ implies that

$$\lim_{n \to \infty} \|A^n\|_{\mathcal{B}(\phi_0)}^{\frac{1}{n}} \to 0 \quad \text{as} \quad n \to \infty.$$ 

Owing to Theorem 5.6, we thus have that $\tau(A) = 0$, so that only the origin could possibly reside in $\sigma(A)$. In fact, since by Theorem 5.6 we must have that $\sigma(A) \neq \emptyset$, it follows that indeed $0 \in \sigma(A)$, and so $\sigma(A) = \{0\}$. Finally, we show that $0 \in \sigma_r(A)$ in the same way as in the previous solution method, so that necessarily $\sigma_p(A) = \sigma_c(A) = \emptyset$ and $\sigma(A) = \sigma_r(A)$. □

$$(Ax, y) = (0, \varepsilon_1 x_1 \overline{y_1}, \varepsilon_2 x_2 \overline{y_2}, \ldots) = (0, x_1(\varepsilon_1 y_1), x_2(\varepsilon_2 y_2), \ldots) = (x, Ay).$$

It follows that $A$ is a self-adjoint, bounded operator on the Hilbert space $\ell^2$. By Theorem 5.9, we have that $\tau(A) = \|A\|$, where $\tau(A)$ is the spectral radius of $A$, defined in (5.7) (we note $\sigma(A) \neq \emptyset$ by Theorem 5.6).

We claim that $\|A\| = 0$. First, let $M := \max_{j \in \mathbb{N}} \varepsilon_j$ (which exists since $\varepsilon_j \to 0$ as $j \to \infty$). It is clear that $M > 0$. Now let $\delta \in (0, M)$ be arbitrary. Then there exists
\(J_\delta \in \mathbb{N}, J_\delta \geq 2\) such that for each \(j \geq J_\delta\), we have \(\varepsilon_j < \frac{1}{2}\delta\). Define the sequence \(x = (x_j)_{j=1}^\infty\) by

\[ x_j = \frac{\delta}{2M \sqrt{J_\delta - 1}}, \quad j = 1, \ldots, J_\delta - 1, \quad x_{J_\delta} = \sqrt{1 - \frac{\delta^2}{4M^2}}, \quad x_{j} = 0, j > J_\delta. \]

Since finitely many terms of the sequence are non-zero, it is clear that \(x \in \ell^2\), and moreover

\[ \|x\|_2^2 = \sum_{j=1}^\infty |x_j|^2 = 1. \]

Furthermore,

\[ \|Ax\|_2^2 = \sum_{j=1}^\infty \varepsilon_j^2 |x_j|^2 = \sum_{j=1}^{J_\delta - 1} \varepsilon_j^2 \frac{\delta^2}{4M^2(J_\delta - 1)} + \sum_{j=J_\delta}^\infty \varepsilon_j^2 |x_j|^2 \]

\[ \leq \frac{\delta^2}{4} + \frac{\delta^2}{4} \sum_{j=J_\delta}^\infty \varepsilon_j^2 |x_j|^2 = \frac{\delta^2}{2} < \delta^2, \]

that is,

\[ \|Ax\|_\ell^2 < \delta. \]

Since \(\delta > 0\) was arbitrary in \((0, M)\), it follows that

\[ \|A\| = \inf_{x \in X, \|x\|_1 = 1} \|Ax\|_\ell^2 = 0. \]

Therefore, \(r(A) = 0\), so that at most the origin could reside in \(\sigma(A)\). In fact, since by Theorem 5.6 we must have that \(\sigma(A) \neq \emptyset\), it follows that indeed \(0 \in \sigma(A)\), and so \(\sigma(A) = \{0\}\). Finally, we show that \(0 \in \sigma_c(A)\) in the same way as in the previous solution method, so that necessarily \(\sigma_p(A) = \sigma_c(A) = \emptyset\) and \(\sigma(A) = \sigma_r(A)\). \(\square\)

### 3 Problem 3

Let \(X = L^2(\mathbb{R}^N)\). For some \(j \in \{1, \ldots, N\}\), let \(A\) be defined by

\[ Au = \frac{\partial u}{\partial x_j}, \quad \mathcal{D}(A) = \left\{ u \in X : \frac{\partial u}{\partial x_j} \in X \right\}, \]

where the distributional derivative is used. Prove that \(A\) is a closed operator and find all three parts of \(\sigma(A)\).
Solution. Method 1. We first prove that $A$ is a closed operator on $X$. We need to prove that the set
\[ G_A = \{(x, y) : x \in \mathcal{D}(A), y = Ax\} \]
is closed in $X \times X$. Let \( \{u_n\} \subset \mathcal{D}(A) \) and \( u, v \in X \) such that \( u_n \to u \) in $X$ and \( Au_n \to v \) in $X$. For each \( \phi \in C_c^\infty(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \), observe the calculation
\[
-(u, \partial_j \phi) = \lim_{n \to \infty} -(u_n, \partial_j \phi) = \lim_{n \to \infty} (\partial_j u_n, \phi) = \lim_{n \to \infty} (Au_n, \phi) = (v, \phi), \tag{3.1}
\]
where \( (\cdot, \cdot) = \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^N)} \) is the inner product on $L^2(\mathbb{R}^N)$. It follows that $v$ is the distributional $x_j$-derivative of $u$, and since $v \in X$, we have $u \in \mathcal{D}(A)$ and $Au = v$. Thus $A$ is closed.

Let $\mathcal{F} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ denote the Fourier Transform on $L^2(\mathbb{R}^N)$. It is known that $\mathcal{F}$ is an isometric isomorphism on $L^2(\mathbb{R}^N)$. Fix $\lambda \in \mathbb{C}$. For each $u \in L^2(\mathbb{R}^N)$, write $\hat{u} = \mathcal{F}(u)$ and note that
\[
\mathcal{F}((\lambda - A)u)(\xi) = \lambda \hat{u}(\xi) - i \xi_j \hat{u}(\xi) = (\lambda - i \xi_j) \hat{u}(\xi), \quad \text{for all } \xi \in \mathbb{R}^N, \xi = (\xi_1, \ldots, \xi_N). \tag{3.2}
\]
Denote $\varphi(\xi) := i \xi_j$, and $M = M_\varphi$ is the maximal multiplication operator associated to $\varphi$ as in Definition 5.10. Note that $\mathcal{R}(\varphi)$ is the imaginary axis $I := \{z \in \mathbb{C} : \text{Re } z = 0\}$, which is a closed set. For each $z \in I$,\[
\varphi^{-1}(\{z\}) = \{\xi \in \mathbb{R}^N : \xi_j = z\},
\]
which is a linear subspace of $\mathbb{R}^N$ with dimension $N - 1$, and so it has $0 \text{-th}$ dimensional Lebesgue measure. By Theorem 5.11, we thus have that $\sigma_r(M) = \sigma_p(A) = \emptyset$, and $\sigma_c(M) = I$. Since $\mathcal{F}$ is an isometric isomorphism on $L^2(\mathbb{R}^N)$, (3.2) implies that
\[
\lambda - M = \mathcal{F}(\lambda - A)\mathcal{F}^{-1}. \tag{3.3}
\]
Hence $\lambda - A$ is injective if and only if $\lambda - M$ is injective, $\mathcal{R}(\lambda - A)$ is dense in $X$ if and only if $\mathcal{R}(\lambda - M)$ is dense in $X$, and $(\lambda - A)^{-1}$ is bounded if and only if $(\lambda - M)^{-1}$ is bounded. It follows that $\sigma_\ell(A) = \sigma_\ell(M)$ for $\ell = p, c, r$. Thus $\sigma_r(A) = \sigma_p(A) = \emptyset$, and $\sigma(A) = \sigma_c(A) = I$. \hfill \Box

Method 2. (Note to grader: please grade only method 1) Although the above proof using the properties of the Fourier Transform is very elegant and straightforward, we sought a “brute force” solution, to see if we could directly characterize the spectrum of $A$. In what follows, we have a complete proof of the characterization of the spectrum of $A$, which does not use any knowledge about the Fourier Transform. This exercise turned out to be very difficult, but quite enlightening, so it is shown here for the reader’s pleasure.

First we prove the injectivity. This is equivalent to showing that if $u \in \mathcal{D}(A)$ and $(\lambda - A)u = 0$, then $u = 0 \in X$. So, if $u \in \mathcal{D}(A)$ satisfies $(\lambda - A)u = 0$, then it is easy to show that
\[
(\text{Re } \lambda)|u|^2 = \text{Re} \left( \frac{\partial u}{\partial x_j} \bar{u} \right) = \frac{1}{2} \frac{\partial |u|^2}{\partial x_j}. \tag{3.4}
\]
The above calculation shows that $|u|^2$ is monotone as a function on $\xi_j$ (this makes sense, because the fact that $u \in \mathcal{D}(A)$ implies $|u|^2 \in L^1(\mathbb{R}^N)$ and also $\partial_j|u|^2 \in L^1$, so that $|u|^2$ is absolutely continuous on almost every line where we fix all coordinates $x_k, k \neq j$). But no non-zero monotone function on $\mathbb{R}$ can belong to $L^1(\mathbb{R})$ (simply because it would be strongly away from 0 on a set of infinite measure), so on almost every line, $|u|^2 \equiv 0$. Hence $|u|^2 = 0$ a.e. on $\mathbb{R}^N$, which implies that $u \equiv 0 \in X$.

We now prove that $\mathcal{R}(\lambda - A)$ is dense in $L^2(\mathbb{R}^N)$ for each $\lambda \in \mathbb{C} \setminus I$, where $I := \{ z \in \mathbb{C} : \Re z = 0 \}$. Let $f \in C^\infty_c(\mathbb{R}^N)$, and write $(y, t) \in \mathbb{R}^N$, $y \in \mathbb{R}^{N-1}$, $t \in \mathbb{R}$, and for each $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we have $x = (y, t)$ with $x_j = t$. Consider the functions

$$u_1(y, t) = \int_t^\infty e^\lambda(t-s) f(y, s) \, ds,$$

and

$$u_2(y, t) = - \int_{-\infty}^t e^\lambda(t-s) f(y, s) \, ds.$$

If $\Re \lambda > 0$, then write $u \equiv u_1$. If $\Re \lambda < 0$, then write $u \equiv u_2$. It is a straightforward calculation that $\lambda u(x) - \partial_j u(x) = f(x)$ for each $x \in \mathbb{R}^N$. We have to prove that $u \in L^2(\mathbb{R}^N)$. This comes for free from the fact that $f$ is bounded with compact support, and the functions $u_1, u_2$ have the desired exponential decay properties on the $t-$variable. Finally, let us prove that $(\lambda - A)^{-1}$ is bounded on $L^2(\mathbb{R}^N)$. Suppose $\Re \lambda > 0$ and let $f \in C^\infty_c(\mathbb{R}^N)$. Then $(\lambda - A)^{-1} f = u_1$. Note that $|e^{\lambda s}| = e^{(\Re \lambda)s}$, for each $s \in \mathbb{R}$. Using the Cauchy-Schwartz inequality, we have that

$$\|u\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^{N-1}} \int_{-\infty}^\infty \left| \int_t^\infty e^{\lambda(t-s)} f(y, s) \, ds \right|^2 \, dt \, dy \leq \int_{\mathbb{R}^{N-1}} \int_{-\infty}^\infty \left( \int_t^\infty |e^{2\lambda s}| |e^{-\lambda s}| |f(y, s)|^2 \, ds \right) \left( \int_t^\infty |e^{-\lambda s}| \, ds \right) \, dt \, dy \leq \int_{\mathbb{R}^{N-1}} \int_{-\infty}^\infty \left( \int_t^\infty |e^{2\lambda s}| |e^{-\lambda s}| |f(y, s)|^2 \, ds \right) \frac{1}{\Re \lambda} e^{-(\Re \lambda) t} \, dt \, dy \leq \frac{1}{\Re \lambda} \int_{\mathbb{R}^{N-1}} \int_{-\infty}^\infty \int_t^\infty e^{(\Re \lambda)(t-s)} |f(y, s)|^2 \, ds \, dt \, dy = \frac{1}{\Re \lambda} \int_{\mathbb{R}^{N-1}} \int_{-\infty}^\infty \int_{-\infty}^s e^{(\Re \lambda)(t-s)} |f(y, s)|^2 \, dt \, ds \, dy \leq \frac{1}{(\Re \lambda)^2} \int_{\mathbb{R}^{N-1}} \int_{-\infty}^\infty |f(y, s)|^2 \, ds \, dy = \frac{1}{(\Re \lambda)^2} \|f\|_{L^2(\mathbb{R}^N)}, \quad (3.6)$$
as desired (we remark that the order in which the inequalities are used is very important here, as the estimate is delicate: note that using the Minkowski Inequality at the beginning to Fubini the right-hand side quickly, or using the Cauchy-Schwartz inequality in a different way will yield useless estimates). The proof when \( \Re \lambda < 0 \) is analogous; in that case we use \( u = u_2 \).

We now prepare to show that \( \mathcal{R}(\lambda - A) \) is dense in \( X \) for \( \lambda \in I \). A first idea would be to try to prove that \( \mathcal{R}(\lambda - A) = C_c^\infty(\mathbb{R}^N) \), but this is not true, because in general \( (\lambda - A)^{-1} f \) may not exist in \( L^2(\mathbb{R}^N) \), even if it exists “locally”. A fix for this technical difficulty comes from the following

**Theorem 3.7.** Fix \( \beta \in \mathbb{R} \). The set

\[
K_\beta := \left\{ f \in C_c^\infty(\mathbb{R}) : \int_\mathbb{R} f(s)e^{-i\beta s} \, ds = 0 \right\}
\]

is dense in \( L^2(\mathbb{R}) \).

**Proof.** It is clear that it is enough to show that if \( f \in C_c^\infty(\mathbb{R}) \), then there exists \( \{f_n\}_{n=1}^\infty \subset K_\beta \) such that \( f_n \to f \) in \( L^2(\mathbb{R}) \). So fix \( f \in C_c^\infty(\mathbb{R}) \). Then there exists \( R > 0 \) such that \( \text{supp } f \subset (-R, R) \). Consider the functions

\[
f_n(t) = f(t) - \frac{1}{n} \left( \int_\mathbb{R} f(s)e^{-i\beta s} \, ds \right) e^{i\beta t} \chi_{(R+1,R+n+1)}(t).
\]

Observe that \( \int_\mathbb{R} f_n(t)e^{-i\beta t} \, dt = 0 \); outside of \((-R, R)\) we have \( f_n \to 0 \), and

\[
\int_{\mathbb{R} \setminus (-R,R)} |f_n|^2 = \left( \int_\mathbb{R} f(s)e^{-i\beta s} \, ds \right)^2 \frac{1}{n} \to 0 \text{ as } n \to \infty,
\]

so that \( f_n \to f \) in \( L^2(\mathbb{R}) \). After mollification with a smooth, nonnegative, compactly supported kernel, it is clear that we may preserve the above properties, and obtain that \( \{f_n\} \subset K_\beta \). This finishes the proof of the theorem (interestingly, \( K_\beta \) is not dense in \( L^1(\mathbb{R}) \)). \( \square \)

Let \( \lambda = i\beta \), where \( \beta \in \mathbb{R} \), and fix \( f \in C_c^\infty(\mathbb{R}^N) \) such that \( f(y, \cdot) \in K_\beta \) for each \( y \in \mathbb{R}^{N-1} \). Then simply choose \( (\lambda - A)^{-1} f := u_2 \). Since \( f(y, \cdot) \in K_\beta \) for each \( y \in \mathbb{R}^{N-1} \) and \( f \) is compactly supported on \( \mathbb{R}^N \), we have that \( u_2 \in X \). But \( \lambda - A \) is unbounded in \( X \): If \( \lambda = 0 \), then it is known that \( (\lambda - A)^{-1} \) is unbounded. If \( \lambda \neq 0 \), then if \( (\lambda - A)^{-1} \) were bounded, then it would imply that \( \|\partial_j u\|_X \leq C\|f\| \), but this is known to be false. \( \square \)

### 4 Problem 4

Let \( X = L^1(0, \infty) \) and

\[
(A u)(x) = \frac{1}{x^2} u \left( \frac{1}{x} \right) \quad (u \in X, x \in (0, \infty)).
\]
We claim that

\[ \text{Convergence Theorem implies that } Au \ \text{since } u \ \text{as functions on } [\frac{1}{N}, N] \text{ exists, and} \]

\[ \text{Consider } \chi_{(\frac{1}{N}, N)}(x) \text{ such that } v \text{ satisfies (4.3). If } v \in L^1(0, \infty), \text{then } \chi_{(\frac{1}{N}, N)}(x) \text{ is bounded and compactly supported on } (0, \infty), \text{and therefore } v \in X. \]

\[ \text{We verify that } v \text{ satisfies (4.3). If } x \in [\frac{1}{2}, 2], \text{then } \]

\[ v(x) = \frac{1}{x} = \frac{1}{x^2}x = \frac{1}{x^2}v \left( \frac{1}{x} \right), \]  

while if \( x \in (0, \infty) \setminus [\frac{1}{2}, 2] \), then \( v(x) = v \left( \frac{1}{x} \right) = 0 \) and so (4.3) is satisfied for such \( x \).

Hence \( v - Av = 0 \) but \( v \) differs from 0 on the interval \( [\frac{1}{2}, 2] \) so it is not identically 0. Hence \( 1 \in \sigma_p(A) \). Now let \( \lambda = -1 \). In this case we will exhibit a function \( u \in X, u \) not identically 0 in \( X \), such that

\[ u(x) = -\frac{1}{x^2}u \left( \frac{1}{x} \right), \]  

for a.e. \( x \in (0, \infty) \).

Consider \( \chi_{(\frac{1}{4}, 1)} = \chi_{[\frac{1}{4}, 1)} - \chi_{(1, 2]} \). If \( x \in (0, \infty) \setminus [\frac{1}{2}, 2] \), then (4.4) is satisfied trivially at such \( x \). If \( x \in [\frac{1}{2}, 1) \), then

\[ v(x) = \frac{1}{x} = -\frac{1}{x^2}(-x) = -\frac{1}{x^2}v \left( \frac{1}{x} \right). \]

Find all three parts of \( \sigma(A) \).

**Solution.** For any \( u \in X \), it is clear that the function \((Au)(x)\) is measurable, since \( \frac{1}{x}, \frac{1}{x^2} \) are continuous in \((0, \infty)\) and \( u \) is measurable. Fix \( N \in \mathbb{N}, N \geq 2 \). Then the map \( f : [\frac{1}{N}, N] \to [\frac{1}{N}, N] \) given by \( f(x) = \frac{1}{x} \) is a bijective, continuously differentiable function, such that its inverse map is also continuously differentiable (in fact, \( f^{-1} = f \) as functions on \([\frac{1}{N}, N]\)). Therefore, we can use the Change of Variables Theorem to see that

\[ \int_{[\frac{1}{N}, N]} |Au(x)| \, dx = \int_{[\frac{1}{N}, N]} \frac{1}{x^2} |u \left( \frac{1}{x} \right)| \, dx = \int_{[\frac{1}{N}, N]} |u(y)| \, dy, \]  

for each \( N \in \mathbb{N} \) (here, \( dx \) is the restriction of the 1-dimensional Lebesgue measure to the interval \([\frac{1}{N}, N]\)). Consequently, the limit as \( N \to \infty \) of the expression on the left-hand side of (4.1) exists, and

\[ \lim_{N \to \infty} \int_{[\frac{1}{N}, N]} |Au(x)| \, dx = \|u\|_X, \]  

since \( u \in X = L^1(0, \infty) \). Since \( |Au(x)| \geq 0 \) a.e. on \((0, \infty)\), the Lebesgue Monotone Convergence Theorem implies that \( Au \in X \), and moreover,

\[ \|Au\|_X = \|u\|_X. \]  

We claim that \( \{1, -1\} \subset \sigma_p(A) \). Let \( \lambda = 1 \). Then \( \lambda \in \sigma_p(A) \) if and only if there exists \( u \in L^1(0, \infty), u \) is not identically 0 in \( X \), such that \( \lambda u - Au = 0 \), or equivalently,

\[ u(x) = \frac{1}{x}u \left( \frac{1}{x} \right), \]  

for a.e. \( x \in (0, \infty) \). For any \( x \in X \),

\[ \chi_{(\frac{1}{N}, N)}(x) \text{ satisfies (4.3). If } x \in [\frac{1}{2}, 2], \text{then } v(x) = v \left( \frac{1}{x} \right) = 0 \]  

and so (4.3) is satisfied for such \( x \).

Hence \( v - Av = 0 \) but \( v \) differs from 0 on the interval \([\frac{1}{2}, 2]\) so it is not identically 0. Hence \( 1 \in \sigma_p(A) \). Now let \( \lambda = -1 \). In this case we will exhibit a function \( u \in X, u \) not identically 0 in \( X \), such that

\[ u(x) = -\frac{1}{x^2}u \left( \frac{1}{x} \right), \]  

for a.e. \( x \in (0, \infty) \).

Consider \( v(x) = \frac{1}{x} \chi_{(\frac{1}{4}, 1)}(x) - \chi_{(1, 2]}(x) \). If \( x \in (0, \infty) \setminus [\frac{1}{2}, 2] \), then (4.4) is satisfied trivially at such \( x \). If \( x \in [\frac{1}{2}, 1) \), then

\[ v(x) = \frac{1}{x} = -\frac{1}{x^2}(-x) = -\frac{1}{x^2}v \left( \frac{1}{x} \right). \]
Similarly, (4.4) is also satisfied for \( x \in (1, 2] \). We remark that \( \{1\} \) is a set of measure 0 in \((0, \infty)\), and therefore we have verified that \( \lambda v - Av = 0 \). We conclude that \(-1 \in \sigma_p(A)\).

Now we claim that \( \rho(A) = \mathbb{C} \backslash \{-1, 1\} \). Fix \( \lambda \in \mathbb{C} \backslash \{-1, 1\} \). Let us first show that \( \lambda - A \) is bijective onto \( X \). This is equivalent to proving that for any \( v \in X \), the equation \( (\lambda - A)u = v \) has a unique solution \( u \in X \). Equivalently, we need to prove that for each \( v \in X \), there is a unique function \( u \in X \) such that

\[
\lambda u(x) - \frac{1}{x^2} u\left(\frac{1}{x}\right) = v(x), \quad \text{for a.e. } x \in (0, \infty).
\] (4.5)

Fix \( v \in X \). If there exists a solution \( u \in X \) to (4.5), then for almost every \( x' \in (0, \infty) \), \( u \) must also satisfy (4.5) for \( x = \frac{1}{x'} \). Therefore we conclude that such a solution \( u \) will also satisfy

\[
\lambda u\left(\frac{1}{x}\right) - x^2 u(x) = v\left(\frac{1}{x}\right), \quad \text{for a.e. } x \in (0, \infty).
\] (4.6)

Now, for almost every \( x \in (0, \infty) \), we must have that both \( v(x), v\left(\frac{1}{x}\right) \) are complex numbers. Fix such \( x \in (0, \infty) \), and observe that we may regard (4.5)-(4.6) as a system of two linear equations in two unknowns. More precisely, consider the system

\[
\begin{aligned}
\lambda a - \frac{1}{x^2} b &= v(x) \\
-x^2 a + \lambda b &= v\left(\frac{1}{x}\right),
\end{aligned}
\] (4.7)

Since \( \lambda \in \mathbb{C} \backslash \{-1, 1\} \), we have that

\[
\begin{vmatrix}
\lambda & -\frac{1}{x^2} \\
-x^2 & \lambda
\end{vmatrix} = \lambda^2 - 1 \neq 0.
\]

Consequently, elementary linear algebra then tells us that there exist unique values for \( a_x, b_x \) such that (4.7) holds. In fact, we can solve for \( a_x \) to obtain that

\[
a_x = \frac{\lambda}{\lambda^2 - 1} v(x) + \frac{1}{\lambda^2 - 1} \frac{1}{x^2} v\left(\frac{1}{x}\right).
\] (4.8)

Since \( x \in (0, \infty) \) is arbitrary outside of a set of measure 0, then outside of a set of measure 0 in \((0, \infty)\) we can define the complex-valued function \( u \) by \( u(x) = a_x \), where \( a_x \) is given by (4.8) (and on the set of measure 0, simply assign arbitrary values to \( u \)). Then \( u \) satisfies (4.5) for a.e. \( x \in (0, \infty) \). Note that by definition of \( u \), we have that

\[
u(x) = \frac{1}{\lambda^2 - 1} \left[ \lambda v(x) + (Av)(x) \right], \quad \text{for a.e. } x \in (0, \infty).
\] (4.9)

So \( u \) is a sum of measurable functions, hence measurable. Also, \( v, Av \in X \), hence \( u \in X \).

Since \( v \in X \) was arbitrary and \( u \in X \) given by (4.9) satisfies \( (\lambda - A)u = v \), we have proven that \( \mathcal{R}(\lambda - A) \) is surjective onto \( X \). In particular, \( \mathcal{R}(\lambda - A) \) is dense in \( X \).
Actually, we have also proven that \((\lambda - A)\) is injective: given \(v \in X\) and any function \(u' \in X\) satisfying \((\lambda - A)u' = v\), then \(u'\) must satisfy (4.5)-(4.6) for a.e. \(x \in (0, \infty)\). But then, for a.e. \(x \in (0, \infty)\), we must have that \(u'(x) = a_x\), where \(a_x\) is given by (4.8) and is the first component of the unique solution \((a_x, b_x)\) to the system (4.7). Thus \(u'\) and \(u\) agree pointwise a.e. on \((0, \infty)\), so that \(u' = u\) as elements in \(X\).

Therefore, the operator \((\lambda - A)^{-1}\) exists. It remains only to show that \((\lambda - A)^{-1}\) is bounded on \(X\). Fix \(v \in X\), and let \(u = (\lambda - A)^{-1}v \in X\). As we have seen above, \(u\) satisfies (4.9). Therefore,

\[
\|u\|_X \leq \frac{1}{|\lambda|^2 - 1}\left[|\lambda|\|v\|_X + \|Av\|_X\right] = \frac{1}{|\lambda|^2 - 1}\left[|\lambda|\|v\|_X + \|v\|_X\right] = \frac{|\lambda| + 1}{|\lambda|^2 - 1}\|v\|_X,
\]

where we used (4.2). The above estimate shows that \((\lambda - A)^{-1}\) is bounded, whenever \(\lambda \in \mathbb{C}\setminus\{-1, 1\}\). This ends the proof of the claim that \(\rho(A) = \mathbb{C}\setminus\{-1, 1\}\).

Finally, we remark that \(\sigma_c(A) = \sigma_r(A) = \emptyset\), and \(\sigma(A) = \sigma_p(A) = \{-1, 1\} = \mathbb{C}\setminus\rho(A)\). \(\square\)

**Remark 4.10.** Let us give two remarks, regarding intuition behind the solution to this problem:

- When \(|\lambda| \neq 1\), the following quick argument shows that \((\lambda - A)^{-1}\) exists (not necessarily on all of \(X\)) and is bounded: For any \(u \in X\), observe by the Triangle Inequality and (4.2) that

\[
\|\lambda A u\|_X \geq \|\lambda u\|_X - \|A u\|_X = \|\lambda\| \|u\|_X - \|u\|_X = |1 - |\lambda|| \|u\|_X.
\]

The above inequality can be used to show the injectivity of \((\lambda - A)\) into \(X\), and it clearly gives the boundedness of \((\lambda - A)^{-1}\). However, this argument fails for \(\lambda\) in the unit circle centered at the origin, and an argument for why \(\mathcal{B}(\lambda - A)\) is dense in \(X\) is also needed. The strength of the method shown in the solution is that it captures all the desired properties of \((\lambda - A)^{-1}\) in one swoop, for all \(\lambda \in \mathbb{C}\setminus\{-1, 1\}\).

- The map \(x \mapsto \frac{1}{x}\) is a bijection on \((0, \infty)\), which takes \((0, 1)\) to \((1, \infty)\) and viceversa. Moreover, observe that \(AAu = u\), and therefore \(A\) is invertible on \(X\), with \(A^{-1} = A\). It is for this reason that we expected that the function \(u\) solving (4.5) must also satisfy an additional “consistency” equation, namely (4.6), which can be obtained by applying \(A\) to both sides of (4.5).

- Note that \((\lambda - A)(\lambda + A) = \lambda^2 - A^2 = \lambda^2 - 1\). Hence \((\lambda - A)^{-1} = \frac{1}{\sqrt{\lambda - 1}}[\lambda + A]\).

5 Appendix

**Definition 5.1.** Let \(X\) be a complex Banach space, and \(A : X \to X\) a closed linear operator on \(X\).
• We say that $\lambda \in \rho(A) \subset \mathbb{C}$ if the following conditions hold: the operator $\lambda - A$ is injective into $X$, $\mathcal{R}(\lambda - A)$ is dense in $X$, and $(\lambda - A)^{-1}$ is bounded. We call $\rho(A)$ the resolvent set of $A$.

• The set $\sigma := \mathbb{C} \setminus \rho(A)$ is called the spectrum of $A$.

• We say $\lambda \in \sigma_p(A) \subset \sigma(A)$ if $\ker(\lambda - A) \neq \{0\}$. The set $\sigma_p(A)$ is known as the point spectrum of $A$, and its elements are known as eigenvalues. The geometric multiplicity of an eigenvalue is $\dim \ker(\lambda - A)$. The algebraic multiplicity of an eigenvalue is $\dim \left( \bigcup_{m=1}^{\infty} \ker(\lambda - A)^m \right)$.

• The continuous spectrum $\sigma_c(A)$ is the set of $\lambda \in \sigma(A) \setminus \sigma_p(A)$ such that $\mathcal{R}(\lambda - A)$ is dense in $X$ but $(\lambda - A)^{-1}$ is not bounded on $X$.

• The residual spectrum $\sigma_r(A)$ is the set $\sigma(A) \setminus (\sigma_p(A) \cup \sigma_r(A))$. Alternatively, it consists of $\lambda \in \sigma(A)$ for which $\mathcal{R}(\lambda - A)$ is not dense in $X$.

**Proposition 5.2.** If $\lambda \in \rho(A)$, then $\mathcal{R}(\lambda - A) = X$, so that $(\lambda - A)^{-1} \in L(X)$.

Now, for $\lambda \in \rho(A)$, define the $L(X)$-valued function $R(\lambda, A) : \mathbb{C} \to L(X)$ by

$$R(\lambda, A) = (\lambda - A)^{-1} \in L(X).$$

We call $R(\lambda, A)$ the resolvent of $A$.

**Theorem 5.3.** The resolvent set $\rho(A)$ is open in $\mathbb{C}$, and the resolvent is a holomorphic $L(X)$-valued function. For any $\mu, \lambda \in \rho(A)$,

$$R(\lambda, A)R(\mu, A) = R(\mu, A)R(\lambda, A),$$

and

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A).$$

**Theorem 5.6.** Let $A \in L(X)$. Then $\sigma(A) \neq \emptyset$ and it is a compact set. Let

$$r(A) := \max \left\{ |\lambda| : \lambda \in \sigma(A) \right\},$$

which we call the spectral radius of $A$. Then

$$r(A) = \lim_{n \to \infty} \| A^n \|^\frac{1}{n}.$$  

**Theorem 5.9.** If $X$ is a Hilbert space and $A$ is a self-adjoint bounded operator on $X$, then $r(A) = \| A \|$.

**Definition 5.10.** Let $\varphi : \mathbb{R}^N \to \mathbb{C}$ be a continuous function. Then the maximal multiplication operator is the operator $M = M \varphi : \mathcal{D}(M) \to L^2(\mathbb{R}^N)$ such that $\mathcal{D}(M) = \left\{ u \in L^2(\mathbb{R}^N) : \varphi u \in L^2(\mathbb{R}^N) \right\}$, $Mu = \varphi u$. 

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Theorem 5.11. Denote by $\mu$ the Lebesgue Measure on $\mathbb{R}^n$. For any continuous $\varphi : \mathbb{R}^n \to \mathbb{C}$, $M$ is closed, $\sigma(M) = \mathcal{R}(\varphi)$,

$$
\sigma_p(M) = \{ \lambda \in \mathbb{C} : \mu(\varphi^{-1}(\{\lambda\})) > 0 \}, \quad \sigma_c(M) = \{ \lambda \in \mathbb{C} : \mu(\varphi^{-1}(\{\lambda\})) = 0 \},
$$

and $\sigma_r(M) = \emptyset$.

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