Boundary Value Problems for second-order elliptic operators with lower order terms

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This is a presentation on the expository paper [Pog].

Outline of the talk.

• Introduction and motivation.

• Project I: Exponential decay of the fundamental solution to generalized Schrödinger operators.

• Quick review of history of the boundary value problems.

• Project II: A perturbation result.
The homogeneous second-order elliptic operator

Let $N \in \mathbb{N}$, $N \geq 2$, and $\Omega \subseteq \mathbb{R}^N$ open. Write

$$L_0 \equiv -\text{div } A \nabla.$$  \hspace{1cm} (1)

Here, $A = (A_{i,j})$ is an $N \times N$ matrix of complex $L^\infty$ coefficients satisfying the following uniform ellipticity condition

$$\lambda |\xi|^2 \leq \Re \langle A(x)\xi, \xi \rangle \quad \text{and} \quad \|A\|_{L^\infty(\Omega)} \leq \Lambda,$$ \hspace{1cm} (2)

for some $\lambda > 0$, $\Lambda < \infty$, and for all $\xi \in \mathbb{C}^N$, $x \in \Omega$. 

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BVPs for inhomogeneous second-order elliptic operators
The Dirichlet problem

The classical Dirichlet problem:

\[
\begin{cases}
-\text{div} \ A \nabla u = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega.
\end{cases}
\]

(3)

- Before the 1970’s: Consider $\Omega$ with smooth boundary, coefficients, data.

- Beginning in 1977 with Dalbergh’s [Dah77]: Solve (3) on Lipschitz (roug"{e}r) domains, with rough coefficients and data.

- Since 1977: On what domains $\Omega$ can we solve (3)? What $A$ (material)? What $f$ (data)? How does the solution depend on the data $f$?
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- Before the 1970's: Consider \( \Omega \) with smooth boundary, coefficients, data.

- Beginning in 1977 with Dalbergh's [Dah77]: Solve (3) on Lipschitz (rougher) domains, with rough coefficients and data.

- Since 1977: On what domains \( \Omega \) can we solve (3)? What \( A \) (material)? What \( f \) (data)? How does the solution depend on the data \( f \)?
Potentials

\[
\begin{cases}
-\text{div} \ A \nabla u + Vu = 0 & \text{in } \Omega, \\
u = f & \text{on } \partial \Omega.
\end{cases}
\]

(4)

• For what kind of $\Omega, A, f,$ and $V$ can we solve (4)?

• Examples: The classical Schrödinger operator $-\Delta + V$.

• The classical magnetic Schrödinger operator $-(\nabla - i \mathbf{a})^2 + V$. 
The tug-of-war between

- The 2nd order term \(-\text{div} \, A \nabla\)
- Existence of harmonic measure
- Necessary conditions on \(A\) known
- Scale-invariance

Homogeneity of equation

- The potential \(V\)
- Exponential decay of solutions in the presence of confining or disordered (random) potential
- Introduce different scaling

Exponential decay
The Dirichlet problem and harmonic measure

- for $E \subset \partial \Omega$, $X \in \Omega$, $\omega^X(E)$ is a solution to

$$-\Delta u = 0 \quad \text{in } \Omega, \quad u \bigg|_{\partial \Omega} = 1_E$$

evaluated at point $X$, that is, $u(X)$.

- $\omega^X(E)$ is the probability for a Brownian motion starting at $X \in \Omega$ to exit through the set $E \subset \partial \Omega$.

- the solution to $-\Delta u = 0 \quad \text{in } \Omega, \quad u \bigg|_{\partial \Omega} = f$
is realized as $u(X) = \int_{\partial \Omega} f \, d\omega^X$
The big goal

We aim to understand the impact of the potential on solutions of boundary value problems (BVPs).

When $V$ is substantial
- Impact of exponential decay?
- Project I: Estimates on resolvents, fundamental solution. Submitted for publication, [MP].
- Long term: Solve BVPs

When $V$ is small
- In which sense is smallness “innocent”?
- Project II: BVPs
Consider

\[ L_E = -\text{div} \ A\nabla + V, \]  
\[ \text{(electric Schrödinger)} \]

or

\[ L_M = - (\nabla - ia)^2 + V, \]  
\[ \text{(magnetic Schrödinger)} \]

or more generally

\[ L = -(\nabla - ia)^T A(\nabla - ia) + V. \]  
\[ \text{(generalized magnetic Schrödinger)} \]

The fundamental solution \( \Gamma \), \( L\Gamma(x, y) = \delta_x(y) \).

**Question.** What is the sharp rate of exponential decay of \( \Gamma \)?

- Exponential decay of solutions to Schrödinger operators in presence of positive potentials: important in quantum physics.
- However, establishing a precise rate of decay for complicated potentials is a challenging open problem. (Landis conjecture)
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**Question.** What is the sharp rate of exponential decay of \( \Gamma \)?

- Exponential decay of solutions to Schrödinger operators in presence of positive potentials: important in quantum physics.
- However, establishing a precise rate of decay for complicated potentials is a challenging open problem. (Landis conjecture)
• For \( \Gamma \) the fundamental solution to \(-\text{div} \, A \nabla + V\),

\[
\frac{c_1 e^{-\varepsilon_1 d(x,y,V)}}{|x - y|^{n-2}} \leq \Gamma(x, y) \leq \frac{c_2 e^{-\varepsilon_2 d(x,y,V)}}{|x - y|^{n-2}}.
\]

(5)

• Only upper estimate for \(- (\nabla - ia)^2 + V\).

• \( L^2 \) estimate for \(- (\nabla - ia)^T A (\nabla - ia) + V\), and resolvents.
We say that $w \in L^p_{\text{loc}}(\mathbb{R}^n)$, with $w > 0$ a.e., belongs to the Reverse Hölder class $RH_p = RH_p(\mathbb{R}^n)$ if there exists a constant $C$ so that for any ball $B \subset \mathbb{R}^n$, 

$$\left( \int_B w^p \right)^{1/p} \leq C \int_B w. \quad (6)$$
Project I: The Fefferman-Phong-Shen maximal function $m(x, w)$

Denote

$$D_a = \nabla - ia,$$

and the magnetic field by $B$, so that

$$B = \text{curl } a.$$  \hspace{1cm} (7)

For a function $w \in RH_p, p \geq \frac{n}{2}$, define the maximal function $m(x, w)$ by

$$\frac{1}{m(x, w)} := \sup_{r > 0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x, r)} w \leq 1 \right\},$$  \hspace{1cm} (8)

and the Agmon distance

$$d(x, y, w) = \inf_{\gamma} \int_{0}^{1} m(\gamma(t), w)|\gamma'(t)| \, dt,$$  \hspace{1cm} (9)

where $\gamma : [0, 1] \to \mathbb{R}^n$ is absolutely continuous and $\gamma(0) = x, \gamma(1) = y$. 
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where $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ is absolutely continuous and $\gamma(0) = x, \gamma(1) = y$. 
Project I: $m(x, w)$ and the uncertainty principle

The function $m$ measures the sum of the contributions of the kinetic energy $\Re AD_a f \overline{D_a f}$ and potential energy $V|f|^2$, and is related to the uncertainty principle through the Fefferman-Phong inequality:

Suppose that $a \in L^2_{\text{loc}}(\mathbb{R}^n)^n$, and moreover assume (12) (next slide). Then, for all $u \in C^1_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} m^2(x, V + |B|)|u|^2 \, dx \leq C \int_{\mathbb{R}^n} (|D_a u|^2 + V|u|^2) \, dx. \quad (10)$$
Theorem 1 (Mayboroda-P. 2018)

For \( L \equiv - (\nabla - i a)^T A (\nabla - i a) + V \) and \( \forall f \in L^2_c(\mathbb{R}^n) \), \( \exists \) constants \( \tilde{d}, \varepsilon, C > 0 \) such that

\[
\int_{\left\{ x \in \mathbb{R}^n \mid d(x, \text{supp} f, V + |B|) \geq \tilde{d} \right\}} m(\cdot, V + |B|)^2 \left| L^{-1} f \right|^2 e^{2\varepsilon d(\cdot, \text{supp} f, V + |B|)} \leq C \int_{\mathbb{R}^n} |f|^2 \frac{1}{m(x, V + |B|)^2}, \tag{11}
\]

provided

i) either \( a = 0 \) and \( V \in \text{RH}_{n/2} \),

ii) or, more generally, \( a \in L^2_{\text{loc}}(\mathbb{R}^n) \), \( V > 0 \) a.e. on \( \mathbb{R}^n \), and

\[
\begin{cases}
V + |B| \in \text{RH}_{n/2}, \\
0 \leq V \leq c m(\cdot, V + |B|)^2, \\
|\nabla B| \leq c' m(\cdot, V + |B|)^3.
\end{cases} \tag{12}
\]
Theorem 1 (Mayboroda-P. 2018)

For $L \equiv -(\nabla - ia)^T A(\nabla - ia) + V$ and $\forall f \in L^2_c(\mathbb{R}^n)$, there exist constants $\tilde{d}, \varepsilon, C > 0$ such that

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V + |B| \in RH_{n/2}, \\
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\end{cases} \quad (12)
\]
Project 1: $L^2$ exponential decay for the resolvent

An analogous estimate holds for the resolvent operator $(I + t^2 L)^{-1}$, $t > 0$:

$$\int \left\{ x \in \mathbb{R}^n \mid d(x, \text{supp } f, \mathcal{B}_t) \geq \tilde{d} \right\} m\left( \cdot, \mathcal{B}_t \right)^2 \left| (I + t^2 L)^{-1} f \right|^2 e^{2\varepsilon d(\cdot, \text{supp } f, \mathcal{B}_t)} \leq C \int_{\mathbb{R}^n} \left| f \right|^2 m\left( \cdot, \mathcal{B}_t \right)^2 .$$

where $\mathcal{B} := V + |B| + \frac{1}{t^2}$.

- In other words, $L^{-1} f$ decays as $e^{-\varepsilon d(\cdot, \text{supp } f, V + |B|)}$ away from the support of $f$ and the resolvent decays as $e^{-\varepsilon d(\cdot, \text{supp } f, V + |B| + \frac{1}{t^2})}$.
- Previous resolvent decay results purely in terms of $\frac{1}{t^2}$. 
Theorem 2 (Mayboroda-P. 2018)

If, in addition, solutions have pointwise (Moser) estimates, then

\[ |\Gamma(x, y)| \leq C e^{-\varepsilon d(x, y, V + |B|)} \frac{|x - y|^{n-2}}{|x - y|^{n-2}} \quad \text{for all } x, y \in \mathbb{R}^n. \] (13)

- In particular, applies to both electric Schrödinger \(-\text{div } A \nabla + V\), and magnetic Schrödinger \(-(\nabla - ia)^2 + V\).
Theorem 3 (Mayboroda-P. 2018)

If, furthermore, an $m$–scale invariant Harnack inequality holds, then

$$|\Gamma(x, y)| \geq c e^{-\varepsilon_2 d(x, y, V + |B|)} \frac{|x - y|^{n-2}}{|x - y|^n}.$$  \hspace{1cm} (14)

- If $\Gamma_E$ is the fundamental solution to electric Schrödinger:

$$\frac{c_1 e^{-\varepsilon_1 d(x, y, V)}}{|x - y|^{n-2}} \leq \Gamma_E(x, y) \leq \frac{c_2 e^{-\varepsilon_2 d(x, y, V)}}{|x - y|^{n-2}}.$$  \hspace{1cm} (15)
Magnetic Schrödinger exhibits *gauge invariance*: quantitative assumptions should be put on $B$ rather than $a$.

The *diamagnetic inequality*

$$ \left| \nabla |u(x)| \right| \leq \left| D_{\mathbf{a}} u(x) \right|. \quad (16) $$

When $A \equiv I$ so that $L_M := L = (\nabla - i\mathbf{a})^2 + V$, $L_M$ is *dominated* by $L_E := -\Delta + V$: for each $\varepsilon > 0$,

$$ |(L_M + \varepsilon)^{-1} f| \leq (\Delta + \varepsilon)^{-1} |f|, \quad \text{for each } f \in H = L^2(\mathbb{R}^n). \quad (17) $$

The above is known as the *Kato-Simon inequality*. 
• First exp. decay in terms of $V$: Agmon [Agm82], but not sharp.

• Shen in [She99]: Sharp exp. decay of $-\Delta + V$.

• Kurata [Kur00]: non-sharp exp. decay of $L_E, L_M$ through heat kernel estimates.
• $L^2$ Germinet and Klein [GK03]: resolvent exp. decay purely in terms of $\frac{1}{t^2}$. Also Combes-Thomas estimates.

• The estimate (11) (for the operator $L^{-1}$) is entirely new and is a consequence of the decay afforded by our assumptions on $V$ and $B$.

• Resolvent estimate $\Rightarrow$ estimate on $f(L)$ for any holomorphic $f$. • Our results are in the nature of best possible.
• **Existence of fundamental solution.** Some authors ([Ben10], [KS00]) took ad-hoc assumptions on $a, V$. We prove existence of a fund. solution in the natural context, by smooth approximation.

• **Non self-adjointness.** When $A$ is not self-adjoint: difficulty in proving some technical lemmas.

• Dealing with complex coefficients, possible lack of Harnack or even of Moser.
• For eigenfunctions, the decay is governed by the uncertainty principle - see Arnold, David, Filoche, Jerison and Mayboroda ‘Localization of eigenfunctions via an effective potential’ [ADFJM].

• Obtain exponential decay in terms of the landscape potential $1/u$, $Lu = 1$. Use sharper uncertainty principle of [ADFJM].

• Solve boundary value problems.
Let us turn our attention to the general second-order elliptic operators, and describe some history.
Let $N \in \mathbb{N}$, $N \geq 2$, and $\Omega \subseteq \mathbb{R}^N$ open. Write

$$L = -\text{div}(A\nabla + b_1 \cdot) + b_2 \cdot \nabla + V. \quad (18)$$

Here, $A$ is an $N \times N$ matrix satisfying $\exists \lambda > 0, \Lambda < \infty$, $\forall \xi \in \mathbb{C}^N$, $x \in \Omega$,

$$\lambda |\xi|^2 \leq \Re \langle A(x)\xi, \xi \rangle \quad \text{and} \quad \|A\|_{L^\infty(\Omega)} \leq \Lambda. \quad (19)$$

- The terms $b_1, b_2$ are called drift terms, while $V$ is called the potential. In particular, electric and magnetic Schrödinger.
Non-tangential maximal operators

In this generality, Lipschitz domain $\iff$ half-space. Given $q \in \mathbb{R}^n$, write the non-tangential approach region

$$\Gamma(q) := \{ (x, t) \in \mathbb{R}_{+}^{n+1} : |x - q| < t \}. \quad (20)$$

If $u \in L^\infty_{loc}(\mathbb{R}_+^{n+1})$, write

$$N_* u(q) := \sup_{(x, t) \in \Gamma(q)} |u(x, t)|. \quad (21)$$

If $u \in L^2_{loc}(\mathbb{R}_+^{n+1})$, write

$$\tilde{N}_* u(q) := \sup_{(x, t) \in \Gamma(q)} \left( \int_{|x-y| < t} \int_{|t-s| < \frac{t}{2}} |u|^2 \right)^{\frac{1}{2}}. \quad (22)$$

We use the notation $u \rightarrow f$ non-tangentially (or $u \rightarrow f \ n.t.$) to mean that for almost every $q \in \mathbb{R}^n$, we have

$$\lim_{\Gamma(q) \ni x \to q} u(x) = f(q). \quad (23)$$

Bruno Poggi  BVPs for inhomogeneous second-order elliptic operators
The Dirichlet problem \((D_p)\)

The Dirichlet problem with data \(f\) in \(L^p\), \(p \in (1, \infty)\) is

\[
\begin{cases}
Lu = 0 \text{ in } \mathbb{R}^{n+1}, \\
u \rightarrow f \text{ non-tangentially}, \\
\|N_* u\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.
\end{cases}
\]

(24)

- Solvability of \((D_p)\) for some \(p\) \iff \(\omega \in A_\infty\).

- For each \(p\), solvability of \((D_p)\) \iff \(\omega \in RH_{p'}\).
The Neumann problem \((N_p)\) and Regularity problem \((R_p)\)

Neumann \((N_p)\): \[
\begin{cases}
L u = 0 \text{ in } \mathbb{R}^{n+1}_+ , \\
\frac{\partial u}{\partial \nu} = g \in L^p(\mathbb{R}^n) , \\
\|\tilde{N}^* (\nabla u)\|_{L^p(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\mathbb{R}^n)}.
\end{cases}
\]

Regularity \((R_p)\): \[
\begin{cases}
L u = 0 \text{ in } \mathbb{R}^{n+1}_+ , \\
u \rightarrow f \text{ n.t.} , \\
\|\tilde{N}^* (\nabla u)\|_{L^p(\mathbb{R}^n)} \lesssim \|\nabla\| f\|_{L^p(\mathbb{R}^n)}.
\end{cases}
\]
BVPs: Early results

• $L = -\Delta$ on Lipschitz $\Omega$: Dalhberg [Dah77] solved $(D_2)$. Jerison and Kenig [JK81b] $(N_2), (R_2)$. Verchota [Ver84] layer potentials.

• $L = -\text{div} \ A \nabla$, how general can $A$ be? Caffarelli, Fabes and Kenig [CFK81]: some regularity in transverse direction is necessary for $(D_2)$. Thus it is natural to consider $t$–independent $A$. 
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BVPs: $t-$independent $A$

- **A real, symmetric**: Jerison and Kenig [JK81a] ($D_2$), Kenig and Pipher [KP93] ($N_2$), ($R_2$).

- **A real, non-symmetric**: Kenig, Koch, Pipher and Toro [KKPT00], Kenig and Rule [KR09], Barton [Bar13], Hofmann, Kenig, Mayboroda and Pipher [HKMP15a]-[HKMP15b].

- **A complex block**: ($N_2$), ($R_2$) equivalent to solving a Kato square root problem. Full solution in [AHLMT02].
BVPs: \( t \)-independent \( A \)

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BVPs: Perturbation results

• $L^\infty$ perturbations of $t$–independent $A$: Fabes, Jerison and Kenig [FJK84], Alfonseca, Auscher, Axelsson, Hofmann and Kim [AAAHK11], Auscher, Axelsson and Hofmann [AAH08], Auscher, Axelsson and McIntosh [AAM10]...

• Carleson-norm perturbations: Fefferman, Kenig and Pipher [FKP91], Kenig and Pipher [KP93]-[KP95], Auscher and Axelsson [AA11], Hofmann, Mayboroda and Mourgoglou [HMM15]...

• Perturbation results for $t$–independent matrices ⊕ Perturbation results under small Carleson-norm assumption
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- Perturbation results for $t$–independent matrices $\oplus$ Perturbation results under small Carleson-norm assumption
BVPs: Inhomogeneous setting

Very little is known. Only two results:

- Kenig and Pipher [KP01], Dindos, Petermichl and Pipher [DPP07]: $b_1 \equiv V \equiv 0$ and some regularity on $b_2 \implies (D_p)$. Used results of Hofmann and Lewis [HL01]

- Shen [She94]: $A \equiv I$, $b_1 \equiv b_2 \equiv 0$, $V \in RH_\infty \implies (N_p), (R_p)$. 
Conjecture 1

Let $\Omega = \mathbb{R}^{n+1}_+$. Suppose that $A$ is a $t$–independent, complex, elliptic, bounded matrix and

$$\max \left\{ \| b_1 \|_{L^n(\mathbb{R}^n)}, \| b_2 \|_{L^n(\mathbb{R}^n)}, \| V \|_{L^{n \over 2}(\mathbb{R}^n)} \right\} \leq \varepsilon_0 \ll 1. \tag{27}$$

If $(D_2), (N_2), (R_2)$ are solvable for $L_0 := -\text{div} \, A \nabla$ and its adjoint $L_0^*$, then they are solvable for $L$.

- Joint work with S. Bortz, S. Hofmann, J. L. Luna García, and S. Mayboroda.
- In [GHN16]: Solution to a Kato problem for inhomogeneous operators.
- Hypotheses hold for $A$ real and symmetric.
- $L^p$ spaces in (27) are scale-invariant with respect to the second-order term $-\text{div} \, A \nabla$. 
Project II: Some definitions

Denote $2^* = \frac{2n}{n-2}$, define the space

$$\tilde{W}^{1,2}(\mathbb{R}^{n+1}_+) = \left\{ u \mid \nabla u \in L^2(\mathbb{R}^{n+1}_+), \ u(\cdot, t) \in L^{2*}(\mathbb{R}^n) \text{ a.e. } t \right\},$$

Define the single-layer potential, $S : \dot{H}^{-\frac{1}{2}}(\mathbb{R}^n) \rightarrow \tilde{W}^{1,2}(\mathbb{R}^{n+1})$, by

$$S := \left( \text{Tr} \circ (L^{-1})^* \right)^*. \quad (28)$$

The above definition coincides with the classical single-layer potential for the operator $-\Delta$:

$$Sf(x, t) = \int_{\mathbb{R}^n} \Gamma(x, t, y, 0) f(y) \, dy, \quad (x, t) \in \mathbb{R}^{n+1}.$$ 

Define the square function $\mathcal{I}$ acting on functions $f \in L^2(\mathbb{R}^{n+1}_+, \frac{dx \, dt}{t})$:

$$\mathcal{I}f := \|f\| := \left( \iint_{\mathbb{R}^{n+1}_+} |f(x, t)|^2 \, dx \, dt \right)^{\frac{1}{2}}.$$
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$$

Define the **single-layer potential**, $S : \dot{H}^{-\frac{1}{2}}(\mathbb{R}^n) \to \tilde{W}^{1,2}(\mathbb{R}^{n+1}_+)$, by

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Define the **square function** $\mathcal{I}$ acting on functions $f \in L^2 \left( \mathbb{R}^{n+1}_+, \frac{dx \, dt}{t} \right)$:

$$
\mathcal{I} f := \| f \| := \left( \iint_{\mathbb{R}^{n+1}_+} |f(x, t)|^2 \frac{dx \, dt}{t} \right)^{\frac{1}{2}}.
$$
As a main step towards solvability of \((N_2)\), we want to show the estimate

\[
\| \tilde{N}_* (\nabla S f) \|_{L^2(\mathbb{R}^n)} \lesssim \| f \|_{L^2(\mathbb{R}^n)}.
\] (29)

Let us call this estimate the \(N_2\) estimate.
Project II: The plan for \((N_2)\)—solvability

To obtain the \(N_2\) estimate, we want to show the following family of estimates, which together “imply” the \(N_2\) estimate.

- \(\tilde{N}_*(\nabla) < \tilde{N}_*(\partial_t) + \text{Tr}:\)

\[
\|\tilde{N}_*(\nabla S f)\|_{L^2(\mathbb{R}^n)} \lesssim \|\tilde{N}_*(\partial_t S f)\|_{L^2(\mathbb{R}^n)} + \|\nabla\|S f(\cdot, 0)\|_{L^2(\mathbb{R}^n)}. \tag{30}
\]

- \(\text{Tr} < \mathcal{I}:\)

\[
\|\nabla S f(\cdot, 0)\|_{L^2(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}^{n+1}_+} t |\nabla \partial_t S f(x, t)|^2 dx dt \right)^{\frac{1}{2}}, \tag{31}
\]

- The square function estimate:

\[
\||t \partial_t^2 S f|| \lesssim \|f\|_{L^2(\mathbb{R}^n)}. \tag{32}
\]

- The \(\tilde{N}_*(\partial_t)\) estimate:

\[
\|\tilde{N}_*(\partial_t S f)\|_{L^2(\mathbb{R}^n)} \lesssim \left[\|\partial_t S\|_{2 \to 2} + 1\right]\|f\|_2. \tag{33}
\]
Project II: Two done, two more to go.

• The estimates $\tilde{N}_* (\nabla) < \tilde{N}_* (\partial_t) + \text{Tr}, \quad \text{Tr} < \mathcal{S}$ have been shown.

• We aim to achieve the square function estimate by applying a vector-valued $Tb$ theorem (see [GH16]). Need to satisfy the technical hypotheses.

• To get the $\tilde{N}_* (\partial_t)$ estimate, we...?
• Our **first strategy** was to use the methods in [AAAHK11].

• But [AAAHK11] relies heavily on DeGiorgi-Nash-Moser bounds, which guarantee **good decay** properties of the fundamental solution $\Gamma$.

• In our setting, the DeGNM bounds do not hold, and neither do good decay properties of the fundamental solution, if it even exists.

• In [AAAHK11], the $\mathcal{N}(\partial_t)$ estimate is obtained through a Cotlar-type argument using the fine properties of $\partial_t \Gamma$.

• Possible fix: The **first order functional calculus** of [AAH08].
• Our first strategy was to use the methods in [AAAHK11].

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Project II: Some main obstacles, constants are not solutions

• If $L_0 \equiv -\text{div } A \nabla$ and $u$ solves $L_0 u = 0$ on $\Omega$, then for any $c \in \mathbb{C}$, $u - c$ also solves $L_0 u = 0$ on $\Omega$.

• The above property does not hold for the inhomogeneous operators with non-zero terms $b_1, V$.

• Linked to the failure of the DeGNM bounds.

• Needed a new proof method for $N(\nabla) < N(\partial_t) + \text{Tr.}$
Project II: Still to do

- We still need to show the square function estimate and the $\mathcal{N}(\partial_t)$ estimate.
- Overcome massive technical difficulties.
- Obtain the invertibility of the layer potentials.
- Create a plan to obtain the solvability of $(D_2), (R_2)$. 

On the failure of the DeGNM theory at the critical $L^p$ spaces for $b_1, V$

- The DeGiorgi-Nash-Moser theory gives Hölder-regularity for positive solutions to $(-\Delta + V)u = 0$ when $V \in L^{n/2} + \varepsilon$, for $\varepsilon > 0$.

- What happens when $L \equiv -\Delta + V$, $V \in L^{n/2}(\mathbb{R}^n)$?

**Theorem 4**

Let $n \geq 3$, $B$ a ball, and $V \in L^{n/2}(\mathbb{R}^n)$ with arbitrarily small norm. There exists $u \in W^{1,2}_{loc}(B)$ a local weak solution to $Lu = 0$ on $B$, such that $u > 0$, $u \notin L^\infty_{loc}(B)$.

- [KS]: similar result.

- But we proved an arbitrarily high $L^p$—integrability result.
The DeGiorgi-Nash-Moser theory gives Hölder-regularity for positive solutions to \((-\Delta + V)u = 0\) when \(V \in L^{n/2 + \varepsilon}\), for \(\varepsilon > 0\).

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Thanks for listening!

Thank you. :)


References III


