The existence of partially localized periodic-quasiperiodic solutions and related KAM-type results for elliptic equations on the entire space

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Dedicated to the memory of Geneviève Raugel

Abstract

We consider the equation

$$\Delta_x u + u_{yy} + f(u) = 0, \quad x = (x_1, \dots, x_N) \in \mathbb{R}^N,\ y \in \mathbb{R}, \tag{1}$$

where $N \geq 2$ and f is a sufficiently smooth function satisfying $f(0) = 0$, $f'(0) < 0$, and some natural additional conditions. We prove that equation (1) possesses uncountably many positive solutions (disregarding translations) which are radially symmetric in $x' = (x_1, \dots, x_{N-1})$ and decaying as $|x'| \to \infty$, periodic in x_N, and quasiperiodic in y. Related theorems for more general equations are included in our analysis as well. Our method is based on center manifold and KAM-type results.

Key words: Elliptic equations, entire solutions, quasiperiodic solutions, partially localized solutions, center manifold, KAM theorems.

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Contents

1 Introduction

We consider the semilinear elliptic equation

$$\Delta u + u_{yy} + f(u) = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

where $N \geq 2$ and $f : \mathbb{R} \to \mathbb{R}$ is a C^k function, $k \geq 1$, satisfying

$$f(0) = 0, \; f'(0) < 0. \tag{1.2}$$

We generally use the symbol Δ for the Laplace operator in the variables $x = (x_1, \dots, x_N)$, sometimes, when indicated, only with respect to some of these variables. We are particularly interested in the more specific equation

$$\Delta u + u_{yy} - u + u^p = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.3}$$

with $p > 1$.

Equations of the above form, frequently referred to as nonlinear scalar field equations, have been extensively studied from several points of view. Nonnegative solutions, which we focus on in this paper, are often the only meaningful solutions from the modeling viewpoint—thinking of population densities, for example—and also they are the only relevant solutions, playing the role of steady states, in the dynamics of the nonlinear heat equation $u_t = \Delta u + u_{yy} + f(u)$ with positive initial data. In other applications—for example, solitary waves or stationary states of nonlinear Klein-Gordon and Schrödinger equations [4]—finite energy solutions are more relevant.

Best understood among positive solutions of (1.1) are the solutions which are (fully) localized in the sense that they decay to 0 in all variables x, y. A classical result of [24] says that such solutions are radially symmetric and radially decreasing with respect to some center in $\mathbb{R}^{N+1}$. For a large class of nonlinearities, including the nonlinearity in (1.3), it is also known that the localized positive solution is unique, up to translations, see [11, 12, 34, 35, 41, 56]. For general results on the existence and nonexistence of localized positive solutions of (1.1) we refer the reader to [4]. We note that, by Pohozaev's identity, equation (1.3) belongs to the existence class if and only if $p < (N+3)/(N-1)$ [4, 43].

If no decay constraints are imposed, a variety of positive solutions with rather complex structure is known to exist, including saddle-shaped and multiple-end solutions [9, 15, 19, 20, 33] or solutions with infinitely many bumps and/or fronts (transitions) formed along some directions [36, 53]. Such a diverse set of solutions is hardly amenable to any general classification or description. One then naturally tries to understand various smaller classes of solutions characterized by some specific symmetry, periodicity, or decay properties. Similarly as in our previous work, [48], in the present paper we are concerned with solutions with some predetermined structure with respect to the variables $x = (x_1, \ldots, x_N)$, that is, all but one variable y. One can think of solutions which are periodic in $x_1, \ldots, x_N$, localized in $x_1, \ldots, x_N$, or a combination of these two structures. The basic question then is: What can be said about the behavior of such solutions in the remaining variable y?

There is vast literature on solutions which are periodic in all x-variables and in the remaining variable y they exhibit one or multiple homoclinic or heteroclinic transitions between periodic solutions (see [39, 51] and references therein; for related studies of solutions with symmetries instead of the periodicity in the x variables see [3] and references therein).

There is also a number of results concerning positive solutions u localized in all of the x-variables:

$$\lim_{|x|\to\infty} \sup_{y\in\mathbb{R}} u(x,y) = 0. \tag{1.4}$$

Any such solution is likely radially symmetric in x about some center in $\mathbb{R}^N$, cp. [8, 21, 27], although this has not been proved in the full generality yet. As for the behavior in y, solutions that are periodic (and nonconstant) in y were first found in [14] and later, by different methods, in [2, 36]. This has been done for a large class of nonlinearities f, including $f(u) = -u + u^p$ with suitable $p > 1$. (There is much more to the results in [2, 14, 36] than the existence of periodic solutions; for example, certain global branches of such solutions were found in [2, 14]). In [48], we addressed the question whether positive solutions which are quasiperiodic (and not periodic) in y and satisfy (1.4) exist. We proved that this is indeed the case if $N \geq 2$ and the nonlinearity f is chosen suitably. Unfortunately, for a reason that we explain below, the method used in [48] is not applicable in some important specific equations, such as (1.3). The existence of y-quasiperiodic solutions satisfying (1.4) for such equations is an open problem which we find very interesting, but will not address here. Note, however, that our results in the present paper do yield y-quasiperiodic positive solutions of (1.3), albeit they have a different structure in terms of the behavior in x.

The structure of solutions that we examine in this paper is "midway" between full periodicity and full decay in x: the solutions are periodic in some of the x-variables and decay in all the others (this is why we need to assume $N \geq 2$). For definiteness and simplicity of the exposition, we specifically postulate the following condition on u: writing $x' = (x_1, \ldots, x_{N-1})$,

$$\lim_{|x'|\to\infty} \sup_{x_N, y \in \mathbb{R}} u(x', x_N, y) = 0, \quad u \text{ is periodic in } x_N, \tag{1.5}$$

that is, there is just 1 periodicity variable. Other splits between the decay and periodicity variables can be treated by our method in a similar way.

We are mainly concerned with the existence of positive solutions satisfying (1.5) which are quasiperiodic in y. We prove the existence of such solutions for a fairly general class of equations. Our conditions on f require, in addition to (1.2) and sufficient smoothness, that the $(N-1)$-dimensional problem

$$\Delta u + f(u) = 0, \quad x' \in \mathbb{R}^{N-1}, \tag{1.6}$$

possesses a ground state which is nondegenerate and has Morse index 1. Let us recall the meaning of these concepts. By a *ground state* of (1.6) we mean a positive fully localized solution of (1.6). From [24] we know that any ground state u^* of (1.6) is radially symmetric, possibly after a shift in $\mathbb{R}^{N-1}$, so we can write $u^* = u^*(r)$, $r = |x'|$. Consider now the Schrödinger operator $A(u^*) = -\Delta - f'(u^*(r))$, viewed as a self-adjoint operator on $L^2_{\text{rad}}(\mathbb{R}^{N-1})$, the space consisting of all radial $L^2(\mathbb{R}^{N-1})$-functions. Its domain is $H^2(\mathbb{R}^{N-1}) \cap L^2_{\text{rad}}(\mathbb{R}^{N-1})$. Since the potential $f'(u^*(r))$ has the limit $f'(u^*(\infty)) = f'(0) < 0$, the essential spectrum of $A(u^*)$ is contained in $[-f'(0), \infty)$ (cp. [52]). So the condition $f'(0) < 0$ implies that the spectrum in $(-\infty, 0]$ consists of a finite number of isolated eigenvalues; these eigenvalues are all simple due to the radial symmetry. We say that the ground state u^* is *nondegenerate* if 0 is not an eigenvalue of $A(u^*)$. The *Morse index* of u^* is defined as the number of negative eigenvalues of $A(u^*)$. By a well known instability result, the Morse index of any ground state is always at least one.

The two conditions, the nondegeneracy and the Morse index equal to 1, are usually satisfied in equations which have a unique ground state, up to translations (see [11, 12, 34, 35, 41, 56]). A typical example is equation (1.6) with $f(u) = -u + u^p$ if $p > 1$ is Sobolev-subcritical in dimension $N - 1$:

$$p < (N+1)/(N-3)_+ = \begin{cases} (N+1)/(N-3) & \text{if } N > 3, \\ \infty & \text{if } N \in \{2, 3\}. \end{cases}$$

The subcriticality condition is necessary and sufficient for the existence of a ground state of (1.6), see [4]. The uniqueness and the other stated properties of the ground state are proved in [34]. Thus our result applies to equation (1.3) in the subcritical case whenever $f(u) = -u + u^p$ meets our regularity requirement, which is the case if p is an integer or if it is large enough. If $N = 2$, the ground state of the one-dimensional problem (1.6) is nondegenerate, if it exists, and has Morse index 1 for any f satisfying

(1.2). For $N > 2$ and general nonlinearities satisfying (1.2), if ground states on $\mathbb{R}^{N-1}$ exist, it is not necessarily true that all of them have Morse index 1 (see [13, 16, 44]). However, under rather general conditions on f, one can find a ground state with this property as a mountain-pass critical point of the associated energy functional (see [13, 30]). The nondegeneracy condition is not guaranteed in general either, but it is not difficult to show that it holds "generically" with respect to f (cp. [14, Section 4]).

Thus, in comparison with our previous results in [48], the theorems of the present paper apply to a much wider class of nonlinearities f in homogeneous equations (1.1), although, again, the present results deal with a different class of quasiperiodic solutions than [48].

As in [48], our method of proving the existence of quasiperiodic solutions has its grounding in our earlier work [46, 47]. It builds on spatial dynamics and center manifold techniques for elliptic equations (see [32] for the origins of this method, and, for example, [10, 17, 22, 23, 26, 28, 37, 38, 42, 45, 59] and references therein for further developments) and KAM-type results in a finite-differentiability setting. We remark that related results can be found in [54, 58], where quasiperiodic solutions for elliptic equations on the strip in $\mathbb{R}^2$ have been found. The center manifold techniques allow us to relate a class of solutions of the elliptic problem to solutions of a finite-dimensional Hamiltonian system, where the variable y plays the role of time. This is an important step before an application of KAM results, as the original elliptic equation itself is not a well-posed evolution problem when y is viewed as time. Different approaches to partial differential equations which are ill-posed, from the KAM perspective, can be found in [18, 54].

In general terms, our method consists in the following. We consider equations of the form

$$\Delta u + u_{yy} + a(x)u + f_1(x,u) = 0, \quad (x,y) \in \mathbb{R}^N \times \mathbb{R} = \mathbb{R}^{N+1}, \tag{1.7}$$

where $f_1(x,u) = u^2 g(x,u)$ and all the listed functions are sufficiently smooth. The Schrödinger operator $-(\Delta + a(x))$ considered on a suitable space of functions of $x \in \mathbb{R}^N$—the space reflects the structure of the solutions one looks for, cp. (1.4) or (1.5)—is assumed to have $n \geq 2$ negative eigenvalues, all simple, with the rest of its spectrum located in the positive half-line. An application of the center-manifold theorem shows that equation (1.7) admits a class of solutions comprising a finite dimensional manifold. These solutions are in one-to-one correspondence with solutions of an ordinary differential equation (ODE) on $\mathbb{R}^{2n}$, the *reduced equation*, in which the variable y plays the role of time. The reduced equation has a Hamiltonian structure and after a sequence of transformations—a Darboux transformation, a normal form procedure, and action-angle variables—it can be written in a neighborhood of the origin as a small perturbation of an integrable Hamiltonian system. The main issue in applying a suitable KAM theorem is then the verification of a nondegeneracy condition for the integrable Hamiltonian system.

In [48], where we examined solutions localized in all x-variables, we proved that for suitable nonlinearities $f = f(u)$ all the above requirements are satisfied by the functions $a(x) = f'(\varphi(x))$, $f_1(x,u) = f(\varphi(x)+u) - a(x)u$, where φ is a ground state of the equation

$$\Delta u + f(u) = 0, \quad x \in \mathbb{R}^N. \tag{1.8}$$

This way we have proved the existence of positive y-quasiperiodic solutions of (1.1) satisfying (1.4). Now, when $a(x)$ in (1.7) is obtained by the linearization at the ground state, the assumption that the operator $-(\Delta + a(x))$ on $L^2(\mathbb{R}^N)$ has two negative eigenvalues is of utmost importance. Equivalently stated, the assumption requires the ground state φ to have Morse index greater than 1. As mentioned above in connection with the $(N-1)$-dimensional problem (1.6), for many nonlinearities, including $f(u) = u^p - u$, it is known that no such ground state can exist. Examples of nonlinearities f for which a ground state of (1.8) has Morse index greater than 1 do exist, however (see [13, 16, 44]), and to some of those the results of [48] apply.

In our present quest, seeking y-quasiperiodic solutions satisfying (1.5), we choose $a(x) = f'(\varphi(x))$ as the linearization at a ground state φ of the equation $\Delta u + f(u) = 0$ in $\mathbb{R}^{N-1}$, rather than $\mathbb{R}^N$. Viewing φ as a function on $\mathbb{R}^N$ constant in x_N, we consider the operator $-(\Delta + a(x))$ on a suitable space of functions periodic in x_N. In this setting, it is relatively easy, even for $f(u) = u^p - u$, to arrange that $-(\Delta + a(x))$ has two negative eigenvalues by means of a suitable scaling. Applying then the general scheme described above, we obtain a Hamiltonian reduced equation in a form suitable for an application of theorems from the KAM theory. Here we quickly run into a difficulty, and a major difference from [48]: the integrable part of this Hamiltonian is necessarily degenerate. This is due to the symmetries in the problem, regardless of the choice of the nonlinearity $f = f(u)$. To deal with this difficulty, we use KAM type results for Hamiltonian systems with "external parameters" as given in [7, 29]. It turns out that a scaling parameter which we introduce in (1.1) and which plays the role of an external parameter in the reduced Hamiltonian gives us enough control over the linear part of the Hamiltonian for the KAM type results to apply. This new technique is quite flexible, and it is mainly on that account that we are able to prove our results for a large class of equations (1.1), including in particular (1.3) for some values of p.

We formulate our main result, Theorem 2.1, on the existence of y-quasiperiodic solutions satisfying (1.5) in the next section. In the same section, we also state two other new theorems, Theorem 2.3 and 2.5, concerning elliptic equations with parameters. Section 3 contains the proof of Theorem 2.3, which after minor modifications also gives the proof of Theorem 2.5. We will later show how (1.1) can be put in the context of such equations by introducing a scaling parameter and thus derive Theorem 2.1 from Theorem 2.3 (see Section 4).

2 Statement of the main results

In this section, we first introduce some terminology and notation, then state our main results.

Given integers $n \geq 2$, $k \geq 1$, a vector $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ is said to be *nonresonant up to order* k if

$$\omega \cdot \alpha \neq 0 \text{ for all } \alpha \in \mathbb{Z}^n \setminus \{0\} \text{ such that } \ |\alpha| \leq k. \tag{2.1}$$

Here $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$, and $\omega \cdot \alpha$ is the usual dot product. If (2.1) holds for all $k = 1, 2, \dots$, we say that ω is *nonresonant*, or, equivalently, that the numbers $\omega_1, \dots, \omega_n$ are *rationally independent*.

A function $u : (x, y) \mapsto u(x, y) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is said to be *quasiperiodic* in y if there exist an integer $n \geq 2$, a nonresonant vector $\omega^* = (\omega_1^*, \dots, \omega_n^*) \in \mathbb{R}^n$, and an injective function U defined on $\mathbb{T}^n$ (the n-dimensional torus) with values in the space of real-valued functions on $\mathbb{R}^N$ such that

$$u(x, y) = U(\omega_1^* y, \dots, \omega_n^* y)(x) \quad (x \in \mathbb{R}^N,\ y \in \mathbb{R}). \tag{2.2}$$

The vector ω^* is called a *frequency vector* and its components the *frequencies* of u. Obviously, there are always countably many frequency vectors of a given quasiperiodic function, and translations (in x or in y) of quasiperiodic functions are quasiperiodic with the same frequencies.

We emphasize that the nonresonance of the frequency vector is a part of our definition. In particular, a quasiperiodic function is not periodic and, if it has some regularity properties, its image is dense in an n-dimensional manifold diffeomorphic to $\mathbb{T}^n$.

We formulate the following hypotheses on the function $f : \mathbb{R} \to \mathbb{R}$.

(S) $f \in C^\ell(\mathbb{R})$, for some integer $\ell > 15 + N/2$, and $f(0) = 0 > f'(0)$.

(G) Equation (1.6) has a nondegenerate ground state φ of Morse index 1.

It is well known that the decay of φ to zero as $|x'| \to \infty$ is exponential and φ is radial about some center in $\mathbb{R}^{N-1}$ (see [24]). Choosing a suitable translation, we will always assume that it is radially symmetric about the origin. We will often view φ as a function of $x \in \mathbb{R}^N$ independent of the last variable x_N.

Our main result reads as follows.

Theorem 2.1. *Assume that $N \geq 2$ and* (S), (G) *hold. Then there exists an uncountable family of positive solutions of equation* (1.1) *satisfying* (1.5) *such that each of these solutions is radially symmetric in x', even in x_N, and quasiperiodic in y with two (rationally independent) frequencies. The frequency vectors of these quasiperiodic solutions form an uncountable set in $\mathbb{R}^2$.*

Remark 2.2. (i) Our proof shows that the family of solutions as in Theorem 2.1 can be found in any given uniform neighborhood of φ; see Remark 2.4(iii) below. Note, however, that we cannot guarantee that all these solutions have the same period in x_N; see Remark 2.4(ii) for an explanation of this.

(ii) As mentioned in the introduction, our theorem applies to equation (1.3) if $p < (N+1)/(N-3)_+$ is an integer or is sufficiently large. Specifically, if p is not an integer, for hypothesis (S) to be satisfied it is sufficient that $p > 15 + N/2$. Note that exponents p satisfying both relations $15 + N/2 < p < (N+1)/(N-3)_+$ exist only if $N \leq 3$. Integers $p > 1$ satisfying $p < (N+1)/(N-3)_+$ exist if $N \leq 6$. We remark that the smoothness in (S) is just a technical, and by no means optimal, requirement.

Although the values of $f(u)$ for $u < 0$ are irrelevant for the statement of Theorem 2.1, it will be convenient to assume that

$$f(u) > 0 \quad (u < 0). \tag{2.3}$$

In view of the conditions $f(0) = 0 > f'(0)$, this can be arranged, without affecting the smoothness of f, by modifying f in $(-\infty, 0)$.

We will show that Theorem 2.1 is a consequence of a more general theorem dealing with the equation depending on a parameter $s \in \mathbb{R}^d$, $s \approx 0$:

$$\Delta u + u_{yy} + a(x; s)u + f_1(x, u; s) = 0, \quad x \in \mathbb{R}^N,\ y \in \mathbb{R}. \tag{2.4}$$

Here f_1 is a nonlinearity satisfying

$$f_1(x, 0; s) = \frac{\partial}{\partial u} f_1(x, u; s)\Big|_{u=0} = 0 \quad (x \in \mathbb{R}^N,\ s \approx 0), \tag{2.5}$$

and the functions a, f_1 are assumed to be radially symmetric in x', and even and 2π-periodic in x_N. To indicate the 2π-periodicity in x_N, we usually consider a, $f_1(\cdot, u)$ as functions on $\mathbb{R}^{N-1} \times S$, with $S = \mathbb{R} \mod 2\pi$. We formulate the precise hypotheses on a, g shortly, after introducing some notation.

We denote by $C_{\mathrm{b}}(\mathbb{R}^N)$ the space of all continuous bounded (real-valued) functions on $\mathbb{R}^N$ and by $C^k_{\mathrm{b}}(\mathbb{R}^N)$ the space of functions on $\mathbb{R}^N$ with continuous bounded derivatives up to order k, $k \in \mathbb{N} := \{0, 1, 2, \dots\}$. The spaces $C_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S)$ and $C^k_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S)$ are the subspaces of $C_{\mathrm{b}}(\mathbb{R}^N)$ and $C^k_{\mathrm{b}}(\mathbb{R}^N)$, respectively, consisting of the functions which are radially symmetric in x', and 2π-periodic and even in x_N. When needed, we assume that $C_{\mathrm{b}}(\mathbb{R}^N)$, $C^k_{\mathrm{b}}(\mathbb{R}^N)$ are equipped with the usual norms and take the induced norms on the subspaces. For $k \in \mathbb{N}$, the spaces $L^2_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S)$ and $H^k_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S)$ are the closed subspaces of $L^2(\mathbb{R}^{N-1} \times S)$ and $H^k(\mathbb{R}^{N-1} \times S)$, respectively, consisting of all functions which are radially symmetric in x' and even in x_N. We assume the standard norms on (the real spaces) $L^2(\mathbb{R}^{N-1} \times S)$ and $H^k(\mathbb{R}^{N-1} \times S)$—for example, for $v \in L^2(\mathbb{R}^{N-1} \times S)$, $\|v\|^2$ is the integral of v^2 over $\mathbb{R}^{N-1} \times (-\pi, \pi)$—and take the induced norms on the subspaces.

Fix integers $n > 1$ (for the number of frequencies of quasiperiodicity) and $d \geq n-1$ (for the dimension of the parameter space), and let B be an open neighborhood of the origin in $\mathbb{R}^d$. We assume that the functions a and f_1 satisfy the following hypotheses with some integers

$$K > 4n + 1, \quad m > \frac{N}{2}. \tag{2.6}$$

(S1) $a(\cdot; s) \in C^{m+1}_{\text{rad,e}}(\mathbb{R}^{N-1} \times S)$ for each $s \in B$, and the map $s \in B \mapsto a(\cdot; s) \in C^{m+1}_{\text{rad,e}}(\mathbb{R}^{N-1} \times S)$ is of class C^{K+1}.

(S2) $f_1 \in C^{K+m+4}(\mathbb{R}^{N-1} \times S \times \mathbb{R} \times B)$, and for all $\vartheta > 0$ the function f_1 is bounded on $\mathbb{R}^{N-1} \times S \times [-\vartheta, \vartheta] \times B$ together with all its partial derivatives up to order $K+m+4$. Also, (2.5) holds and $f_1(x, u; s)$ is radially symmetric in x' and even in x_N.

The next hypotheses concern the Schrödinger operator $A_1(s) := -\Delta - a(x; s)$ acting on $L^2_{\text{rad,e}}(\mathbb{R}^{N-1} \times S)$ with domain $H^2_{\text{rad,e}}(\mathbb{R}^{N-1} \times S)$.

(A1)(a) There exists $L < 0$ such that

$$\limsup_{|x'| \to \infty} a(x', x_N; s) \leq L, \text{ uniformly in } x_N,\ s.$$

(A1)(b) For all $s \in B$, $A_1(s)$ has exactly n nonpositive eigenvalues,

$$\mu_1(s) < \mu_2(s) < \cdots < \mu_n(s),$$

all of them simple, and $\mu_n(s) < 0$.

Hypotheses (A1)(a) and (A1)(b) will sometimes be collectively referred to as (A1). Hypothesis (A1)(a) guarantees that for all s the essential spectrum $\sigma_{ess}(A_1(s))$ is contained in $[-L, \infty)$ [14, 52]. Since $-L > 0$, hypothesis (S1) and the simplicity of the eigenvalues in (A1)(b) imply that $\mu_1(s), \ldots, \mu_n(s)$ are C^{K+1} functions of s (see [31]). This justifies the use of the derivative in our last hypothesis (ND). Let $\omega(s) := (\omega_1(s), \ldots, \omega_n(s))^T$ (so $\omega(s)$ is a column vector), where

$$\omega_j(s) := \sqrt{|\mu_j(s)|}, \quad j = 1, \ldots, n. \tag{2.7}$$

(ND) The $n \times (d+1)$ matrix $\left[\, \nabla\omega(0) \ \ \omega(0) \,\right]$ has rank n.

We can now state our theorem concerning (2.4).

Theorem 2.3. *Suppose that hypotheses* (S1), (S2) *(with K, m as in* (2.6)*),* (A1), *and* (ND) *are satisfied. Then there is an uncountable set $W \subset \mathbb{R}^n$ consisting of rationally independent vectors, no two of them being linearly dependent, such that for every $(\bar\omega_1, \ldots, \bar\omega_n) \in W$ the following holds: equation* (2.4) *has for some $s \in B$ a solution u such that* (1.5) *holds, and $u(x, y)$ is radially symmetric in x', even and 2π-periodic in x_N, and quasiperiodic in y with frequencies $\bar\omega_1, \ldots, \bar\omega_n$.*

Remark 2.4. (i) Similarly as theorems in [46, 47], Theorem 2.3 gives sufficient conditions in terms of the coefficients and nonlinearities in a given elliptic equation, presently equation (2.4), for the existence of solutions quasiperiodic in y and satisfying required decay and/or symmetry conditions in x. The conclusions of the results in [46, 47] are in some sense stronger: they yield uncountably many quasiperiodic solutions for *every* value of the parameter in a certain range (which may be required to be small enough). In contrast, Theorem 2.3 yields quasiperiodic solutions for *some* values of $s \in B$, possibly leaving out a large set of other values. On the other hand, the present theorem has a weaker nondegeneracy condition than the theorems in [46, 47]. The nondegeneracy conditions in [46, 47] involve some nonlinear terms (quadratic or cubic) in the equation, whereas our present nondegeneracy condition, (ND), is a condition on the coefficient a in the linear part of the equation alone. This makes (ND) much easier to use in applications. Indeed, while the nondegeneracy conditions involving nonlinear terms are "generic" if the class of admissible nonlinearities is large enough, their verification in specific equations, such as the spatially homogeneous equation (1.1), presents a substantial technical hurdle (cp. [48]). The verification of the present condition (ND) is, in principle, simpler; it amounts to showing that one has "good enough" control over the eigenvalues of a linearized problem when parameters are varied.

(ii) When applying Theorem 2.3 in the proof of Theorem 2.1, we introduce a parameter $s \in \mathbb{R}$ in (1.1)—so (1.1) can be viewed in the context of (2.4)—by scaling the variables (x, y). Therefore, the y-quasiperiodic solutions which we find using Theorem 2.3 for some values of s will in fact yield, after the inverse rescaling, y-quasiperiodic solutions of the same original equation (1.1) and, due to the properties of the set W, the frequencies of these quasiperiodic solutions will form an uncountable set. Note, however, that the rescaling changes the period in x_N. This is why we are not able to prescribe the period, say 2π, for the solutions u in Theorem 2.1, with a fixed nonlinearity f.

(iii) The conclusion of Theorem 2.3 (as well as the conclusion of Theorem 2.5 below) remains valid if the solutions u are in addition required to be small in the sense that for an arbitrarily given $\epsilon > 0$ one has $\sup_{(x,y)\in\mathbb{R}^{N+1}} |u(x, y)| < \epsilon$. This follows from the proof, where the solutions are found on a local center manifold of (2.4). Accordingly, for any $\epsilon > 0$ one can find a solution u as in Theorem 2.1 with the property that $\sup_{(x',x_N,y)\in\mathbb{R}^{N+1}} |u(x', x_n, y) - \varphi(x')| < \epsilon$, where φ is the ground state as in (G).

(iv) Evenness with respect to x_N can be dropped in the assumptions on a and g, and in the definition of the domain and the target space of the operator $A_1(s) = -\Delta - a(x; s)$ (and then it has to be dropped in the conclusion of Theorem 2.3). Note, however, that if a, g are even—as will be the case in an application of Theorem 2.3 below—the eigenvalues $\mu_2(s), \dots, \mu_n(s)$ of the operator $-\Delta - a(x; s)$ may be simple in the space of even functions but not in the full space. Similarly, it is possible to drop the assumption of radial symmetry in x', but the simplicity of the eigenvalues may fail to hold in the full space.

(v) A nondegeneracy condition of the same form as (ND) appears in Scheurle's paper [55] on bifurcations of quasiperiodic solutions in analytic reversible ODEs. He used techniques similar to [55] in the paper [54], already mentioned in the introduction, on (analytic) elliptic equations on the strip $\{(x, y) : x \in (0, 1), y \in \mathbb{R}\}$.

The localized-periodic setting in which we consider equation (2.4) reflects our goal to study solutions satisfying (1.5). However, our present techniques can be used in other settings; for example, one can consider a different split between periodicity and decay variables in $x_1, \dots, x_N$. Straightforward, mostly notational, modifications of the arguments below apply in any such setting. As an illustration, we formulate a theorem analogous to Theorem 2.3 in but one different setting: the symmetry and decay (and no periodicity) in all variables x.

We need the following spaces: $C_{\text{rad}}(\mathbb{R}^N)$, $C^k_{\text{rad}}(\mathbb{R}^N)$ consist of all radially symmetric functions in $C_{\text{b}}(\mathbb{R}^N)$ and $C^k_{\text{b}}(\mathbb{R}^N)$, respectively; $L^2_{\text{rad}}(\mathbb{R}^N)$ is the space of all radial $L^2(\mathbb{R}^N)$-functions, and for $k \in \mathbb{N}$, $H^k_{\text{rad}}(\mathbb{R}^N) := H^k(\mathbb{R}^N) \cap L^2_{\text{rad}}(\mathbb{R}^N)$ is the space of all radial $H^k(\mathbb{R}^N)$-functions.

Theorem 2.5. *Let K and m be as in* (2.6)*. Assume that hypotheses* (S1)*,* (S2)*,* (A1)*,* (ND) *are satisfied with $C^{m+1}_{\text{rad,e}}(\mathbb{R}^{N-1} \times S)$ replaced by $C^{m+1}_{\text{rad}}(\mathbb{R}^N)$, $C^{K+m+4}(\mathbb{R}^{N-1} \times S \times \mathbb{R} \times B)$ by $C^{K+m+4}(\mathbb{R}^N \times \mathbb{R} \times B)$, $L^2_{\text{rad,e}}(\mathbb{R}^{N-1} \times S)$ by $L^2_{\text{rad}}(\mathbb{R}^N)$, and $H^2_{\text{rad,e}}(\mathbb{R}^{N-1} \times S)$ by $H^2_{\text{rad}}(\mathbb{R}^N)$; and the last assumption in* (S2) *(radial symmetry in x' and periodicity in x_n) replaced by the assumption that f_1 is radially symmetric in x. Then there is an uncountable set $W \subset \mathbb{R}^n$ consisting of rationally independent vectors, no two of them being linearly dependent, such that for every $(\bar{\omega}_1, \dots, \bar{\omega}_n) \in W$ the following holds: equation* (2.4) *has for some $s \in B$ a solution u such that* (1.4) *holds, and $u(x, y)$ is radially symmetric in x and quasiperiodic in y with frequencies $\bar{\omega}_1, \dots, \bar{\omega}_n$.*

For the proof of this theorem, one just needs to make obvious changes in the proof of Theorem 2.3 consisting mostly of replacements of the underlying spaces as in the formulation of the theorem.

Remark 2.6. If one considers periodicity in two or more variables (say, $(x_1, \dots, x_j)$), the dependence of a and f_1 on those variables may also impose some additional restrictions on the setting, for instance, if a_1 and f do not depend on $(x_1, \dots, x_j)$, then the corresponding periods must be chosen suitably to keep the simplicity of the eigenvalues of $-\Delta - a(x; s)$.

3 Proof of Theorem 2.3

We use the notation introduced in the previous section and assume hypotheses (S1), (S2), (A1), (ND) to be satisfied. Let $B_\delta := \{s \in \mathbb{R}^d : |s| < \delta\}$, where we take $\delta > 0$ so that $B_\delta \subset B$ (below we will make $\delta > 0$ smaller several times).

For $s \in B_\delta$ and $j = 1, \dots, n$, we denote by $\varphi_j(\cdot; s)$ an eigenfunction of the operator $A_1(s)$ associated with the eigenvalue $\mu_j(s)$ normalized in the L^2-norm. For the *principal eigenfunction* $\varphi_1(\cdot; s)$, we may assume that it is positive which determines it uniquely, and it is then of class C^{K+1} as a $H^2_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S)$-valued function of s (see [31]). The same applies to $\varphi_j(\cdot; s)$, provided it is chosen suitably (the normalization determines it uniquely up to a sign). Since $\mu_1(s) < \cdots < \mu_n(s)$ are simple isolated eigenvalues of $A_1(s)$, the eigenfunctions $\varphi_1(\cdot; s), \dots, \varphi_n(\cdot; s)$ have exponential decay as $|x'| \to \infty$ [1, 52].

Since the essential spectrum of $A_1(s)$ is contained in $[-L, \infty)$, the eigenvalues in $(-\infty, -L)$ are isolated in $\sigma(A_1(s))$ and hypotheses (A1)(a), (A1)(b) imply that there is $\gamma > 0$ such that $(0, \gamma) \cap \sigma(A_1(s)) = \emptyset$ for all $s \in B_\delta$.

Hypotheses (S1), (S2), (A1)(a), (A2)(b), (NR) are analogous to some hypotheses in our previous papers [46, 47]. In those papers we mainly focused on solutions which are radially symmetric and decaying in all variables x and, accordingly, the assumptions on the functions a, f_1 involved radial symmetry in x. In the present setting, we assume radial symmetry in x' and periodicity in x_N. As noted in [46, Remark 2.1(v)], [47, Remark 2.1(ii)], the general technical results from [46, 47] apply in the present setting with straightforward modifications of the proofs. In the next subsection, we recall the needed results from [46, 47].

3.1 Center manifold and the structure of the reduced equation

Here we essentially just reproduce Section 3 of [47] (which in turn is an extension of results in Sections 3 and 4 of [46]) with minor adjustments in the notation on the account of the present periodicity-decay setting. The fact that $s \in B_\delta \subset \mathbb{R}^d$, whereas in [47] we had $s \in (-\delta, \delta) \subset \mathbb{R}$, makes no nontrivial difference in the proofs.

We begin with the center manifold reduction. For that we first write equation (2.4) in an abstract form, using the spaces $X := H^{m+1}_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S) \times H^{m}_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S)$, and $Z := H^{m+2}_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S) \times H^{m+1}_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S)$. Let f_1 be as in (2.5). Its Nemytskii operator $\tilde{f} : H^{m+2}_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S) \times B_\delta \to H^{m+1}_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S)$ is given by

$$\tilde{f}(u; s)(x) = f_1(x, u(x); s),$$

and it a well defined map of class C^{K+1} (see [46, Theorem A.1(b)]). The abstract form of (2.4) is

$$\begin{aligned} \frac{du_1}{dy} &= u_2, \\ \frac{du_2}{dy} &= A_1(s)u_1 - \tilde{f}(u_1; s). \end{aligned} \tag{3.1}$$

We rewrite this further as

$$\frac{du}{dy} = A(s)u + R(u; s), \tag{3.2}$$

where $u = (u_1, u_2)^T$,

$$\begin{aligned} A(s)u &= (u_2, A_1(s)u_1)^T, \\ R(u; s) &= (0, \tilde{f}(u_1; s))^T. \end{aligned} \tag{3.3}$$

Here, for each $s \in B_\delta$, $A(s)$ is considered as an operator on X with domain $D(A(s)) = Z$, and R as a C^{K+1}-map from $Z \times B_\delta$ to Z. The notion of a solution of (3.2) on an interval $\mathcal{I}$ is as in [28, 59]: it is a function in $C^1(\mathcal{I}, X) \cap C(\mathcal{I}, Z)$ satisfying (3.2).

Recall that $\varphi_j(\cdot; s)$, $j = 1, \dots, n$, are the eigenfunctions of $A_1(s) := -\Delta - a(x; s)$ corresponding to the eigenvalues $\mu_1(s), \dots, \mu_n(s)$, and they have been chosen so that they are of class C^{K+1} as $H^2_{\text{rad,e}}(\mathbb{R}^{N-1} \times S)$-valued functions of s. By elliptic regularity, for $j = 1, \dots, n$, $\varphi_j(\cdot; s) \in H^{m+2}_{\text{rad,e}}(\mathbb{R}^{N-1} \times S)$ and it is of class C^{K+1} as a $H^{m+2}_{\text{rad,e}}(\mathbb{R}^{N-1} \times S)$-valued function of s. Define the space

$$X_c(s) := \left\{ (h, \tilde{h})^T : h, \tilde{h} \in \operatorname{span}\{\varphi_1(\cdot; s), \dots, \varphi_n(\cdot; s)\} \right\} \subset Z,$$

the orthogonal projection operator

$$\Pi(s) : L^2_{\text{rad,e}}(\mathbb{R}^{N-1} \times S) \to \operatorname{span}\{\varphi_1(\cdot; s), \dots, \varphi_n(\cdot; s)\},$$

and let $P_c(s) : X \to X_c(s)$ be given by $P_c(s)(v_1, v_2) = (\Pi(s)v_1, \Pi(s)v_2)$. As shown in [46, Section 3.2], $P_c(s)$ is the spectral projection for the operator $A(s)$ associated with the spectral set $\{\pm i\omega_j(s) : j = 1, \dots, n\}$ (with $\omega_j(s)$ as in (2.7))—the spectrum of $A(s)$ is the union of this set and a set which is at a positive distance from the imaginary axis. The smoothness of the maps $s \mapsto \varphi_j(\cdot; s)$ implies that $s \mapsto P_c(s)$ is of class C^{K+1} as an $\mathscr{L}(X, Z)$-valued map on B_δ.

Also define $P_h(s) = I_X - P_c(s)$, I_X being the identity map on X, and, for $j = 1, \dots, n$,

$$\psi_j(\cdot; s) = (\varphi_j(\cdot; s), 0)^T, \quad \zeta_j(\cdot; s) = (0, \varphi_j(\cdot; s))^T. \tag{3.4}$$

A basis of $X_c(s)$ is given by

$$\mathscr{B}(s) := \{\psi_1(\cdot; s), \dots, \psi_n(\cdot; s), \zeta_1(\cdot; s), \dots, \zeta_n(\cdot; s)\}.$$

For $z \in X_c(s)$, we denote by $\{z\}_{\mathscr{B}}$ the coordinates of z with respect to the basis $\mathscr{B}(s)$. Denote further

$$\begin{aligned} \psi(s) &:= (\psi_1(\cdot; s), \dots, \psi_n(\cdot; s)), \\ \zeta(s) &:= (\zeta_1(\cdot; s), \dots, \zeta_n(\cdot; s)). \end{aligned} \tag{3.5}$$

The following result is a part of [47, Proposition 3.1], adjusted to the present setting.

Proposition 3.1. *Using the above notation, the following statement is valid, possibly after making $\delta > 0$ smaller. There exist a map $\sigma : (\xi, \eta; s) \in \mathbb{R}^{2n} \times B_\delta \mapsto \sigma(\xi, \eta; s) \in Z$ of class C^{K+1} and a neighborhood $\mathscr{N}$ of 0 in Z such that for each $s \in B_\delta$ one has*

$$\sigma(\xi, \eta; s) \in P_h(s)Z \quad ((\xi, \eta) \in \mathbb{R}^{2n}), \tag{3.6}$$

$$\sigma(0, 0; s) = 0, \quad D_{(\xi,\eta)}\sigma(0, 0; s) = 0, \tag{3.7}$$

and the manifold

$$W_c(s) = \{\xi \cdot \psi(s) + \eta \cdot \zeta(s) + \sigma(\xi, \eta; s) : (\xi, \eta) = (\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n) \in \mathbb{R}^{2n}\} \subset Z$$

has the following properties:

(a) *If $u(y)$ is a solution of* (3.1) *on $\mathcal{I} = \mathbb{R}$ and $u(y) \in \mathscr{N}$ for all $y \in \mathbb{R}$, then $u(y) \in W_c(s)$ for all $y \in \mathbb{R}$; that is, $W_c(s)$ contains the trajectory of each solution of* (3.1) *which stays in $\mathscr{N}$ for all $y \in \mathbb{R}$.*

(b) *If $z : \mathbb{R} \to X_c(s)$ is a solution of the equation*

$$\frac{dz}{dy} = A(s)\big|_{X_c(s)} z + P_c(s) R(z + \sigma(\{z\}_{\mathscr{B}}; s); s) \tag{3.8}$$

on some interval $\mathcal{I}$, and $u(y) := z(y) + \sigma(\{z(y)\}_{\mathscr{B}}; s) \in \mathscr{N}$ for all $y \in \mathcal{I}$, then $u : \mathcal{I} \to Z$ is a solution of (3.1) *on $\mathcal{I}$.*

In the sequel, $W_c(s)$ is called the *center manifold* and equation (3.8) the *reduced equation*.

Next, we examine the Hamiltonian structure of the reduced equation. For $(u, v) \in Z$ and any fixed $s \in B_\delta$, let

$$H(u, v) = \int_{\mathbb{R}^{N-1} \times S} \left(\frac{-1}{2} |\nabla u(x)|^2 + \frac{1}{2} a(x; s) u^2(x) + F(x, u(x); s) + \frac{1}{2} v^2(x) \right) dx, \tag{3.9}$$

where

$$F(x, u; s) = \int_0^u f_1(x, \vartheta; s) d\vartheta.$$

Equation (3.1) has a formal Hamiltonian structure with respect to the functional H and this structure is inherited in a certain way by the reduced equation. More specifically, denoting by Φ the composition of the maps $(\xi, \eta) \to \sigma(\xi, \eta; s) : \mathbb{R}^{2n} \to Z$ and $H : Z \to \mathbb{R}$, (3.8) is the Hamiltonian system with respect to the Hamiltonian Φ and a certain symplectic structure defined in a neighborhood of $(0, 0) \in \mathbb{R}^{2n}$. This is a consequence of general results of [37]; in [46] we gave a proof, with some additional useful information, using direct explicit computations. We have then transformed the system by performing several coordinate changes. By the first one, we achieve that, near the origin, in the new coordinates (ξ', η') the system is Hamiltonian with respect to (the transformed Hamiltonian) and the standard symplectic form on $\mathbb{R}^{2n}$, $\sum_i \xi'_i \wedge \eta'_i$. The existence of such a local transformation is guaranteed by the Darboux theorem, but in [46] we took some care to keep track of how the symplectic structure and the Darboux transformation depend on the parameters. We showed in particular that the Darboux transformation can be chosen as a C^K map in ξ, η, and s, which is the sum of the identity map on $\mathbb{R}^{2n}$ and terms of order $\mathcal{O}(|(\xi, \eta)|^3)$. In the coordinates

(ξ', η') resulting from such a transformation, the Hamiltonian takes the following form for $(\xi', \eta') \approx (0, 0)$:

$$\Phi(\xi', \eta'; s) = \frac{1}{2} \sum_{j=1}^{n} (-\mu_j(s)(\xi'_j)^2 + (\eta'_j)^2) + \Phi'(\xi', \eta'; s). \tag{3.10}$$

Here, $\mu_j(s)$ are the negative eigenvalues of $A_1(s)$, as above, and Φ' is a function of class C^K in all its arguments and of order $\mathcal{O}(|(\xi', \eta')|^3)$ as $(\xi', \eta') \to (0, 0)$. We remark that the formulas given for Φ in [46, 47] are a bit longer, specifying in particular the cubic terms of Φ, but those more precise expressions are not needed here.

We now make a canonical (that is, symplectic form preserving) linear transformation defined by

$$\xi'_j = \frac{1}{\sqrt{\omega_j(s)}} \xi_j, \quad \eta'_j = \sqrt{\omega_j(s)}\, \eta_j \quad (j = 1, \ldots, n), \tag{3.11}$$

where $\omega_j(s) := \sqrt{|\mu_j(s)|}$, $j = 1, \ldots, n$, are as in (2.7). (The coordinates ξ and η used here are not the same coordinates as in Proposition 3.1.) This transformation puts the quadratic part of Φ in the "normal form:" in the coordinates (ξ, η),

$$\Phi(\xi, \eta; s) := \frac{1}{2} \sum_{j=1}^{n} \omega_j(s)(\xi_j^2 + \eta_j^2) + \hat{\Phi}(\xi, \eta; s), \tag{3.12}$$

where $\hat{\Phi}$ is a function of class C^K and of order $\mathcal{O}(|(\xi, \eta)|^3)$ as $(\xi, \eta) \to (0, 0)$.

Later, we will also use the *action-angle* variables $J = (J_1, \ldots, J_n) \in \mathbb{R}^n$, $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n$. They are defined by

$$(\xi_j, \eta_j) = \sqrt{2J_j}(\cos\theta_j, \sin\theta_j) \tag{3.13}$$

in regions where $J_j = (\bar{\xi}_j^2 + \bar{\eta}_j^2)/2 > 0$ for all $j \in \{1, \ldots, n\}$. In these coordinates, the Hamiltonian Φ in (3.12) takes the form

$$\Phi(\theta, J; s) = \omega(s) \cdot J + \hat{\Phi}(\theta, J; s) \tag{3.14}$$

(with the usual abuse of notation: $\hat{\Phi}(\theta, J; s)$ actually stands for $\Phi(\xi(\theta, J), \eta(\theta, J); s)$). The change of coordinates from (ξ_j, η_j) to (θ, J) is also canonical. In particular, in these coordinates the reduced equation reads as follows:

$$\begin{aligned} \dot{\theta} &= \nabla_J \Phi(\theta, J; s), \\ \dot{J} &= -\nabla_\theta \Phi(\theta, J; s). \end{aligned} \tag{3.15}$$

The above Hamiltonian structure is the structure we use below in the proof of Theorem 2.3. We remark that another structure we could use instead is the reversibility of (3.1): if $(u_1(x, y), u_2(x, y))$ a solution, so is $(u_1(x, -y), -u_2(x, -y)))$. This reversibility structure is also inherited by the reduced equation (see [28, 37]). More specifically, writing the equation as an ODE on $\mathbb{R}^{2n}$, there is a transformation D on $\mathbb{R}^{2n}$ such that D^2 is the identity map on $\mathbb{R}^{2n}$ and D anticommutes with the right-hand side of the ODE. (See Remark 3.5 for additional comments on the reversibility structure).

3.2 KAM-type results for systems with parameters and completion of the proof of Theorem 2.3

To prove Theorem 2.3, we apply a KAM-type result from [7, 29] to the reduced Hamiltonian (3.14). To recall that result, consider, for some positive integers n and d, a Hamiltonian $H : \mathbb{T}^n \times \Omega \times B \to \mathbb{R}$ given by

$$H(\theta, I; s) = H^0(I; s) + H^1(\theta, I; s), \tag{3.16}$$

where $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z}^n)$ is the n-dimensional torus (so $H^1(\theta, I; s)$ is 2π-periodic in $\theta_1, \dots, \theta_n$), and Ω, B are bounded domains in $\mathbb{R}^n$, $\mathbb{R}^d$, respectively; $s \in B$ acts as a parameter. We assume that H^0 is (real) analytic on $\Omega \times B$ and $H^1 : \mathbb{T}^n \times \Omega \times B \to \mathbb{R}$ is of class C^k for some $k \geq 2$.

The Hamiltonian system corresponding to H is

$$\begin{aligned} \dot\theta &= \nabla_I H(\theta, I; s), \\ \dot I &= -\nabla_\theta H(\theta, I; s), \end{aligned} \tag{3.17}$$

and the one corresponding to H^0,

$$\begin{aligned} \dot\theta &= \nabla_I H^0(I; s), \\ \dot I &= 0. \end{aligned} \tag{3.18}$$

We denote by ω^* the *frequency map* of H^0:

$$(I; s) \mapsto \omega^*(I; s) := (\nabla_I H^0(I; s))^T : \Omega \times B \to \mathbb{R}^n. \tag{3.19}$$

Here and below we view the gradient as a row vector, so $\omega^*(I; s)$ is a column vector.

For each $s \in B$, the system (3.18) is completely *integrable*. Its state space is covered by invariant tori $\mathbb{T}^n \times \{I_0\}$, $I_0 \in \Omega$, and any such torus is filled with trajectories of quasiperiodic solutions whenever the vector $\omega^*(I_0; s)$ is nonresonant. As usual, for the persistence of some of these quasiperiodic tori under the perturbation in (3.16), we introduce a class of Diophantine frequencies. A vector $\omega \in \mathbb{R}^n$ is said to be *κ, ν-Diophantine*, for some $\kappa > 0$ and $\nu > n - 1$, if

$$|\omega \cdot \alpha| \geq \kappa |\alpha|^{-\nu} \quad (\alpha \in \mathbb{Z}^n \setminus \{0\}). \tag{3.20}$$

Fixing $\nu > n - 1$ arbitrarily, for any nonempty bounded open set $V \subset \mathbb{R}^n$ and $\kappa > 0$, we define

$$V_\kappa := \{\omega \in V : \operatorname{dist}(\omega, \partial V) \geq \kappa \text{ and } \omega \text{ is } \kappa, \nu\text{-Diophantine}\}. \tag{3.21}$$

It is well known that for small $\kappa > 0$ the Lebesgue measure, $|V_\kappa|$, of V_κ is positive; in fact, $|V \setminus V_\kappa| \to 0$ as $\kappa \searrow 0$.

As a nondegeneracy assumption, we shall require the frequency map

$$\omega^*(I, s) = (\omega_1^*(I, s), \dots, \omega_n^*(I, s))^T$$

to have surjective derivative:

(NDsI) The $n \times (n+d)$ matrix

$$\nabla_{I,s}\omega^*(I,s) = \begin{bmatrix} \nabla_{I,s}\omega_1^*(I,s) \\ \vdots \\ \nabla_{I,s}\omega_n^*(I,s) \end{bmatrix}$$

has rank n for all $(I,s) \in \Omega \times B$.

Note that this assumption implies that the range of ω^*, $V = \omega^*(\Omega \times B)$, is an open set in $\mathbb{R}^n$.

The perturbation term H^1 will be assumed to have a sufficiently small norm C^k-norm $\|H^1\|_{C^k(\mathbb{T}^n \times \Omega \times B)}$ which stands for the smallest upper bound, over $\mathbb{T}^n \times \Omega \times B$, on the moduli of all derivatives of H^1 of orders 0 through k.

Theorem 3.2. *Let H^0, ω^* be as above and $V := \omega^*(\Omega \times B)$. Assume that* (NDsI) *holds and let $\nu > n-1$ be fixed. If $k_0 = k_0(\nu)$ is a sufficiently large integer, then the following statement holds. For every $\kappa > 0$ there is $\vartheta > 0$ such that for an arbitrary C^k-map $H^1 : \mathbb{T}^n \times \Omega \times B \to \mathbb{R}$ with $k \geq k_0$ and $\|H^1\|_{C^k(\mathbb{T}^n \times \Omega \times B)} < \vartheta$ the Hamiltonian $H^0 + H^1$ has the following property. There is a C^1 map*

$$\Psi : \mathbb{T}^n \times \Omega \times B \to \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^d$$

of the form

$$\Psi(\theta, I, s) = (T(\theta, I, s), \Upsilon(I, s)), \quad T(\theta, I, s) \in \mathbb{T}^n \times \mathbb{R}^n, \;\; \Upsilon(I, s) \in \mathbb{R}^d, \tag{3.22}$$

which is a near-identity diffeomorphism onto its image and such that for any $(I_0, s_0) \in \mathbb{T}^n \times \Omega$ with $\omega^*(I_0, s_0) \in V_\kappa$ the manifold

$$\tilde{\mathbb{T}}_{(I_0,s_0)} := \{T(\theta, I_0, s_0) : \theta \in \mathbb{T}^n\} \tag{3.23}$$

is invariant under the flow of (3.17) *with $s = \Upsilon(I_0, s_0)$ and the solution of* (3.17) *with the initial condition $T(\theta_0, \omega^*(I_0, s_0))$, $\theta_0 \in \mathbb{T}^n$, is given by $T(\theta_0 + \omega^*(I_0, s_0)t, \omega^*(I_0, s_0))$, $t \in \mathbb{R}$.*

This is a special case of a theorem from [7]: see Corollary 5.1 and Section 5c in [7] for a version of the theorem for analytic Hamiltonians; the adjustments needed in the proof for finitely differentiable Hamiltonians are indicated in the appendix of [7] (see also [29]; statements of the theorem and related results can also be found in [5, 57]). The theorem is an extension of a result of [49] for a Hamiltonian without parameters (that is, $d = 0$), in which case condition (NDsI) is the same as the Kolmogorov nondegeneracy condition.

Remark 3.3. (i) By saying that Ψ is a near-identity diffeomorphism we mean that the C^1 norm of the difference of Ψ and the identity on $\mathbb{T}^n \times \Omega \times B$ is less than 1. One can additionally say that the norm becomes arbitrarily small as $\vartheta \to 0$.

(ii) Since V_κ consists of nonresonant vectors, the solution

$$t \mapsto T(\theta_0 + \omega^*(I_0, s_0)t, \omega^*(I_0, s_0))$$

is quasiperiodic with the frequency vector $\omega^*(I_0, s_0)$. The set of the frequencies of these solutions, V_κ, has positive measure if κ is sufficiently small.

(iii) Specific estimates as to how large $k_0 = k_0(\tau)$ has to be are available. As noted in [7, Appendix], a sufficient but not optimal condition is $k_0 > 4\nu + 2$. Thus, if a regularity class C^k with $k > 4n - 2$ is given upfront, one can always pick $k_0 \leq k$ and $\nu > n - 1$ so that $k_0 > 4\nu + 2$ and then Theorem 3.2 applies with such choices of ν and k_0. We also remark that the diffeomorphism Ψ is more regular than C^1 and its smoothness increases with k (see [49] for more precise differentiability assumptions on the Hamiltonian and the corresponding regularity properties of the map T in the case $d = 0$).

In our application of Theorem 3.2, we consider a Hamiltonian $G : \mathbb{T}^n \times \Omega \times B \to \mathbb{R}$ given by

$$G(\theta, I; s) = \omega(s) \cdot I + G^1(\theta, I; s), \tag{3.24}$$

where $s \mapsto \omega(s) : B \to \mathbb{R}^n$ is a C^1 map satisfying the following condition.

(NDs) The $n \times (d + 1)$ matrix

$$[\ \nabla\omega(s) \quad \omega(s)\ ]$$

has rank n for all $s \in B$.

Note that this is the type of condition satisfied locally by the frequencies in our elliptic problem, see condition (ND) in Section 2.

We will take the *linear* function $G^0(I; s) = \omega(s) \cdot I$ as the unperturbed integrable Hamiltonian and view G^1 as a small C^k perturbation. We relate the Hamiltonians G and H—and conditions (NDs) and (NDsI)—in the following lemma. In the simplest case, when $s \mapsto \omega(s)$ is analytic and $\nabla\omega(s)$ alone has rank n, we can simply take $H^0 = G^0$. This leads to a very similar setup, with the frequencies serving as parameters, as in [50] where a parametrization by frequencies is used in the proof of a classical KAM theorem (see also [40] for an earlier use of a "parametrization" technique). In other cases, some "tricks" will be used to accommodate G^0 in the setting of Theorem 3.2.

Lemma 3.4. *Fix $\nu > n - 1$ and let $k_0 = k_0(\nu)$ be as in Theorem 3.2. Given any $k \geq k_0$, assume that $s \mapsto \omega(s) : B \to \mathbb{R}^n$ is a C^k map satisfying* (NDs). *Then there is $\vartheta > 0$ such that for an arbitrary C^k-map $G^1 : \mathbb{T}^n \times \Omega \times B \to \mathbb{R}$ with $\|G^1\|_{C^k(\mathbb{T}^n \times \Omega \times B)} < \vartheta$ the Hamiltonian $G := G^0 + G^1$ has the following property. There is an uncountable set $W \subset \mathbb{R}^n$ consisting of rationally independent vectors, no two of them being linearly dependent, such that for every $\bar{\omega} \in W$ the Hamiltonian system*

$$\begin{aligned} \dot{\theta} &= \nabla_I G(\theta, I; s), \\ \dot{I} &= -\nabla_\theta G(\theta, I; s) \end{aligned} \tag{3.25}$$

has for some $s \in B$ a quasiperiodic solution of the form $T_s(\bar{\omega}t)$, $t \in \mathbb{R}$, where $T_s : \mathbb{T}^n \to \mathbb{T}^n \times \Omega$ is a C^1 imbedding of the torus $\mathbb{T}^n$.

Proof. First assume that $\nabla\omega(s)$ has rank n for all $s \in B$ and $s \mapsto \omega(s)$ is analytic. Taking $H^0(I; s) := G^0(I; s) = \omega(s) \cdot I$ for all $I \in \Omega$, $s \in B$, we immediately see that condition (NDsI) is satisfied with $\omega^*(I, s) = \omega(s)$ (cp. (3.19)). Let V be the image of B under the map $s \to \omega(s)$. This is an open set in $\mathbb{R}^n$, hence for $\kappa > 0$ small enough, the set V_κ has positive measure. Fix such κ and let $\vartheta = \vartheta(\kappa)$ be as in Theorem 3.2. We claim that the conclusion of Lemma 3.4 holds with this ϑ. Indeed, if G^1 satisfies the smallness condition, then Theorem 3.2 with $H^1 = G^1$ tells us that the conclusion of Lemma 3.4 regarding (3.25) holds for any $\bar{\omega} \in V_\kappa$: we simply choose s_0 with $\omega(s_0) = \bar{\omega}$ and then, with an arbitrary $I_0 \in \Omega$, take $s = \Upsilon(s_0, I_0)$ and define $T_s := T(\cdot, I_0, s_0)$. So to complete the proof in the present case, we just need find an uncountable subset W of V_κ such no two vectors of W are linearly dependent. Such a set exists because, as V_κ has positive measure, there are uncountably many lines through the origin that intersect V_κ. Thus, we can pick a unique vector from V_κ in any such line to form the set W.

Next, still assuming that $\nabla\omega(s)$ has rank n, we remove the analyticity assumption: $\omega(s)$ is now of class C^k. We make, without loss of generality, a simplifying assumption that $d = n$ and ω is a diffeomorphism of B onto its image V. This can always be achieved by replacing B by a small neighborhood of some arbitrarily fixed $s^0 \in B$ and dropping some "disposable" parameters. More precisely, relabeling the parameters $s_1, \dots, s_d$, we may assume that the matrix

$$[\ \partial_{s_1}\omega(s) \dots \partial_{s_n}\omega(s)\ ]$$

has rank n for all $s \approx s^0$. Then, if $d > n$, we consider only those $s \in B$ whose last $d - n$ components, $s_{n+1}, \dots, s_d$, are fixed and equal to the last $d - n$ components of s^0. Accordingly, we replace B by a neighborhood $\tilde{B}$ of s^0 in the corresponding n-dimensional affine space. With the number of parameters equal to n, the rank condition implies that ω is a diffeomorphism, possibly after the neighborhood $\tilde{B}$ of s^0 is made smaller. Of course, proving the statement of the lemma with B replaced by the smaller set $\tilde{B}$ trivially implies the original statement.

The assumption that $\omega : B \to V$ is a diffeomorphism allows us to reparameterize the problem, using the frequency vectors as parameters, in such a way that the linear integrable part becomes analytic in the parameters. For that we denote by $\upsilon : V \to B$ the inverse to $\omega(s)$; this is a C^k map. Let again $\kappa > 0$ be so small that V_κ has positive measure. Clearly, Theorem 3.2 applies to the integrable Hamiltonian $H^0(I, \bar{\omega}) := \bar{\omega} \cdot I$, $I \in \Omega$, $\bar{\omega} \in V$, and the perturbation $H^1(\theta, I, \bar{\omega}) := G^1(\theta, I, \upsilon(\bar{\omega}))$, provided $G^1 : \mathbb{T}^n \times \Omega \times B \to \mathbb{R}$ has sufficiently small C^k-norm. This implies the conclusion of Lemma 3.4 (we choose a subset $W \subset V_\kappa$ with the required properties as in the first part of the proof). Thus Lemma 3.4 is proved in the case that $\nabla\omega(s)$ has rank n.

Finally, we take on the case of the rank of $\nabla\omega(s)$ being less than n; by (NDs), the rank has to be equal to $n-1$, with the vector $\omega(s)$ outside the range of $\nabla\omega(s)$ for each $s \in B$. We introduce an extra real parameter $\beta \approx 1$, so the parameter set becomes $B \times (1-\epsilon, 1+\epsilon)$ for a small $\epsilon > 0$. Consider the linear integrable Hamiltonian $\tilde{G}^0(I; s, \beta) := \beta\omega(s) \cdot I$ and the perturbation $\tilde{G}^1(I, \theta; s, \beta) := \beta G^1(\theta, I, s)$. Due to (NDs), the gradient matrix

$$\nabla_{s,\beta}(\beta\omega(s)) = [\ \beta\nabla_s\omega(s) \quad \omega(s)\ ]$$

has rank n for all $(s, \beta) \in B \times (1-\epsilon, 1+\epsilon)$ if $\epsilon > 0$ is small enough, which we will henceforth assume.

Thus, the part of the statement of Lemma 3.4 already proved above applies to $\tilde{G}^0$, $\tilde{G}^1$, provided $G^1 : \mathbb{T}^n \times \Omega \times B \to \mathbb{R}$ has sufficiently small C^k-norm. This yields a set $\tilde{W} \subset \mathbb{R}^n$ consisting of rationally independent vectors, no two of them being linearly dependent, such that for every $\bar{\omega} \in \tilde{W}$ the Hamiltonian system

$$\begin{aligned} \dot{\theta} &= \beta\nabla_I G(\theta, I; s), \\ \dot{I} &= -\beta\nabla_\theta G(\theta, I; s), \end{aligned} \tag{3.26}$$

has for some $s \in B$, $\beta \in (1-\epsilon, 1+\epsilon)$ a quasiperiodic solution with frequency vector $\bar{\omega}$. Noting that (3.26) is just (3.25) with rescaled time, we get the desired conclusion for (3.25) with a set W obtained from $\tilde{W}$ by multiplying each element $\bar{\omega} \in \tilde{W}$ by a scalar $\beta = \beta(\bar{\omega}) \approx 1$. The vectors obtained this way are mutually distinct, due to the properties of $\tilde{W}$, so W is still uncountable, and the pairwise linear independence is obviously preserved as well. The lemma is proved. □

We remark that for the matrix $\nabla\omega(s)$ to have rank n, we would need $d \geq n$. Hypothesis (NDs), on the other hand, only requires $d \geq n-1$, which "saves" us one parameter.

We are now ready to complete the proof of Theorem 2.3.

Proof of Theorem 2.3. We return to the Hamiltonian of the reduced equation (see (3.12) and (3.14)). In the coordinates (ξ, η),

$$\Phi(\xi, \eta; s) := \frac{1}{2}\sum_{j=1}^{n} \omega_j(s)(\xi_j^2 + \eta_j^2) + \hat{\Phi}(\xi, \eta; s), \tag{3.27}$$

and in the action-angle variables $J = (J_1, \ldots, J_n) \in \mathbb{R}^n$, $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n$ (cp. (3.13)),

$$\Phi(\theta, J; s) = \omega(s) \cdot J + \hat{\Phi}(\theta, J; s). \tag{3.28}$$

Here, J is taken near the origin and such that $J_j > 0$ for all $j \in \{1, \ldots, n\}$, and $s \in B_\delta \subset \mathbb{R}^d$, for some $\delta > 0$.

Recall that $\hat{\Phi}(\xi, \eta; s)$ is of class C^K on a neighborhood of the origin in $\mathbb{R}^{2n} \times \mathbb{R}^d$ and of order $\mathcal{O}(|(\xi, \eta)|^3)$ as $(\xi, \eta) \to (0, 0)$. Therefore, by Taylor's theorem, $\hat{\Phi}(\xi, \eta; s)$

can be written as the sum of finitely many terms, each of them being the product of a degree-three monomial in ξ, η and a C^{K-3} function of ξ, η, s. The function $\hat{\Phi}(\theta, J; s)$ is obtained from this sum by substituting

$$(\xi_j, \eta_j) = \sqrt{2J_j}(\cos\theta_j, \sin\theta_j) \quad (j = 1, \ldots, n)$$

(which introduces some singular behavior in the derivatives of $\hat{\Phi}(\theta, J; s)$ as $J \to 0$). In these action-angle variables, $\hat{\Phi}$ is of order $\mathcal{O}(|J|^{3/2})$ as $|J| \to 0$.

Recall also that $\omega(s) \in \mathbb{R}^n$ is as in (2.7) and it is of class C^{K+1} as a function of s.

Fix constants $k_0 \leq K-3$ and $\nu > n-1$, k_0 being an integer, such that $k_0 > 4\nu+2$. This is possible due to (2.6). According to Remark 3.3(iii), Theorem 3.2 applies with these choices of ν and k_0. We introduce the scaling $J = \epsilon I$ with $\epsilon \in (0, 1)$, $I \in \Omega$, where

$$\Omega := \{I \in \mathbb{R}^n : \ q \leq I_j \leq 2q \ \ (j = 1, \ldots, n)\} \tag{3.29}$$

and q is some positive constant, which we fix for the rest of the proof. Now define G^0, G^1 on $\mathbb{T}^n \times \Omega \times B_\delta$ by

$$\begin{aligned} G^0(I; s) &:= \omega(s) \cdot I, \\ G^1(\theta, I; s) &:= \frac{1}{\epsilon}\,\hat{\Phi}(\theta, \epsilon I; s), \end{aligned} \tag{3.30}$$

which is legitimate for all sufficiently small $\epsilon > 0$ (below we will make an additional smallness requirement on ϵ). We set $G := G^0 + G^1$.

Observe that $G(\theta, I; s) = \Phi(\theta, \epsilon I; s)/\epsilon$, which is the right Hamiltonian for the rescaled reduced equation (3.15): the Hamiltonian system corresponding to the Hamiltonian G in the standard symplectic form is the same as the system obtained from (3.15) after the substitution $J = \epsilon I$ (and it is of course the same as the Hamiltonian system of Φ with respect to the transformed symplectic form corresponding to the *noncanonical* coordinate transformation $(I, \theta) = (\epsilon J, \theta)$).

We are now going to apply Lemma 3.4 to the Hamiltonian $G = G^0 + G^1$, with $\epsilon > 0$ sufficiently small. Take $k := K - 3 \geq k_0$. The smoothness hypotheses of Lemma 3.4 on $s \to \omega(s)$ and G^1 are then satisfied. Hypothesis (NDs) is verified, possibly after $\delta > 0$ is made smaller, due to hypothesis (ND) in Section 2. It remains to verify that the smallness requirement on G^1 is met if $\epsilon > 0$ is small enough. Consider any derivative $D^\alpha\hat{\Phi}(\theta, J; s)$ of order at most k. Here α is a multiindex in $\mathbb{N}^{2n+d}$. We denote by α_J the total number of derivatives in $D^\alpha\hat{\Phi}$ taken with respect to the J-variables. Using our previous observations on the asymptotic behavior of $\hat{\Phi}$ as $J \to 0$ and taking into account the maximal singularity possibly introduced by differentiating one of the roots $J_1^{1/2}, \ldots, J_n^{1/2}$, we obtain that $D^\alpha\hat{\Phi}(\theta, J; s)$ is of order $|J|^{3/2-\alpha_J}$ as $|J| \to 0$. Therefore, taking the corresponding derivative D^α in the variables $(\theta, I; s)$, we discover that for some constant C_α

$$|D^\alpha_{\theta,I;s}G^1(\theta, I; s)| = \frac{1}{\epsilon}\,\epsilon^{\alpha_J}|D^\alpha_{\theta,J;s}\hat{\Phi}(\theta, \epsilon I; s)| \leq C_\alpha\epsilon^{1/2} \quad ((\theta, I, s) \in \mathbb{T}^n \times \Omega \times B_\delta).$$

This implies that if $\epsilon > 0$ is sufficiently small, the condition $\|G^1\|_{C^k(\mathbb{T}^n \times \Omega \times B)} < \vartheta$ of Lemma 3.4 is satisfied.

Having verified all the hypotheses, and fixing a small enough $\epsilon > 0$, we obtain that the system (3.25) has quasiperiodic solutions with frequencies covering the set W, as stated in Lemma 3.4. The trajectories of these solutions are contained in $\mathbb{T}^n \times \Omega$. Undoing the ϵ-scaling, we obtain quasiperiodic solutions of the reduced equation (3.15) whose trajectories are contained in $\mathbb{T}^n \times \epsilon\Omega$. If so desired, we can adjust $\epsilon > 0$ to guarantee that the trajectories are contained in any given neighborhood of $\mathbb{T}^n \times \{0\}$.

We now reverse the transformations made in Section 3.1, namely, the passage to the action-angle variables, transformation (3.11), and the Darboux transformation, to get back to the reduced equation (3.8). This yields quasiperiodic solutions of (3.8), for the same values of s as in (3.25), whose frequencies vectors cover the same set W. Moreover, we can assume that the trajectories of these solutions are all contained in a small neighborhood of the origin (we may need to adjust $\epsilon > 0$ for this, as noted above). In particular, if z is any of these solutions, then $z(y) \in \mathscr{N}$ for all $y \in \mathbb{R}$, $\mathscr{N}$ being the neighborhood of $0 \in Z$ from Proposition 3.1. Then, by Proposition 3.1(b),

$$U(y) = (U_1(y), U_2(y))^T = z(y) + \sigma(\{z(y)\}_{\mathscr{B}}; s) \in Z$$

is a solution of system (3.1). Letting

$$u(x, y) = U_1(y)(x), \tag{3.31}$$

we obtain a solution of (2.4). This solution is quasiperiodic in y, 2π-periodic and even in x_N, and radially symmetric in x' (the periodicity and symmetry come from the definition of the space Z). The frequencies of the solutions obtained this way still cover the same set W, which has the properties required in Theorem 2.3. It remains to show that each solution $u(x, y)$ obtained this way decays to 0 as $|x'| \to \infty$, uniformly in x_N and y. This is a direct consequence of the fact that the set $\{u(\cdot, y) : y \in \mathbb{R}\}$ is contained in a compact set—continuous image of a torus—in $H^{m+2}_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S)$, with $m > N/2$. □

Remark 3.5. As noted at the end of Section 3.1, the reduced equation is reversible and this structure can be used instead of the Hamiltonian structure in the proof of Theorem 2.3. Theorems for reversible systems analogous to Theorem 3.2 can be found in [5, 6, 57], for example, and a result analogous to our Lemma 3.4 can be derived from those. For analytic reversible systems, Scheurle has proved the existence of quasiperiodic solutions under the same nondegeneracy condition as (NDs), see [55].

4 Proof of Theorem 2.1

Assume the hypotheses of Theorem 2.1 to be satisfied. We derive the conclusion of the theorem from Theorem 2.3 with $n := 2$, $K := 10 > 4n + 1$, $m := \ell - 15 > N/2$, with ℓ as in hypothesis (S). Note that f is of class C^{K+m+5}.

To put equation (1.1) in the form (2.4), we linearize a rescaled equation (1.1) about a ground state. Here we initially follow [14]. Let φ be a (radially symmetric) ground state of (1.6), as in hypothesis (G). As assumed in (G), the operator $-\Delta - f'(\varphi(x'))$ considered on $L^2_{\mathrm{rad}}(\mathbb{R}^{N-1})$ with domain $H^2_{\mathrm{rad}}(\mathbb{R}^{N-1})$ has exactly one nonpositive eigenvalue, further denoted by μ_0, and this eigenvalue is negative and simple. For $\lambda > 0$ set $\varphi^\lambda(x') := \varphi(\sqrt{\lambda}x')$. This is a ground state of the rescaled equation

$$\Delta u + \lambda f(u) = 0, \quad x' \in \mathbb{R}^{N-1}. \tag{4.1}$$

In the following, we view φ^λ as a function of $x \in \mathbb{R}^N$, independent of x_N. Set

$$a^\lambda(x) := \lambda f'(\varphi^\lambda(x)).$$

We examine the Schrödinger operator $A^\lambda := -\Delta - a^\lambda(x)$ acting on $L^2_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S)$ with domain $H^2_{\mathrm{rad,e}}(\mathbb{R}^{N-1} \times S)$. The function a^λ has the limit $\lambda f'(0)$ as $|x'| \to \infty$, which is negative due to hypothesis (S). As noted in Section 2, this implies that the essential spectrum of A^λ is contained in $[-\lambda f'(0), \infty)$. Scaling and separation of variables show, as in [14], that the following statements hold. The principal (minimal) eigenvalue of A^λ is $\lambda\mu_0 < 0$ with eigenfunction independent of x_N, and it is a simple eigenvalue. If λ is greater than but close to $-1/\mu_0 > 0$, then the second eigenvalue is $\lambda\mu_0 + 1 < 0$ with eigenfunction of the form $\varsigma(|x'|)\cos x_N$ and it is also a simple eigenvalue. All other eigenvalues (as well as the essential spectrum) of A^λ are positive. Fix any $\lambda > -1/\mu_0$, $\lambda \approx -1/\mu_0$, with these properties and set

$$a(x; s) := a^{\lambda+s}(x) = (\lambda + s) f'(\varphi^{\lambda+s}(x)), \tag{4.2}$$

$$f_1(x, u; s) := (\lambda + s) f(\varphi^{\lambda+s}(x) + u) - a(x; s)u. \tag{4.3}$$

Here $s \in (-\delta, \delta) =: B$, where we take $\delta \in (0, \lambda)$ so small that for all $s \in [-\delta, \delta]$

$$\mu_1(s) := (\lambda + s)\mu_0 < \mu_2(s) := (\lambda + s)\mu_0 + 1 < 0 \tag{4.4}$$

and $\mu_1(s)$, $\mu_2(s)$ are the only nonpositive eigenvalues of $-\Delta - a(x; s)$. Thus, the function $a(x; s)$ satisfies hypotheses (A1)(a) (with $L := (\lambda - \delta) f'(0)$) and (A2)(b) (with $n = 2$).

Obviously, f_1 satisfies (2.5), and the symmetry requirements in (S1), (S2) follow from the definitions of a, f_1, and the symmetry of $\varphi^{\lambda+s}(x') = \varphi(x'(\lambda+s)^{1/2})$. The verification of the smoothness requirements in (S1), (S2), with $d = 1$, is straightforward (and is left to the reader) when one uses the following claim: φ is of class C^{K+m+5} and all its derivatives up to order $K + m + 5$ decay exponentially as $|x'| \to \infty$. To prove this claim, we first note that, since f is of class C^{K+m+4}, the fact that φ is of class C^{K+m+5} (with locally Hölder derivatives of order $K + m + 5$) is a standard elliptic regularity result. Now, since $\varphi(x')$—and consequently $f(\varphi(x'))$— decays exponentially, the equation $\Delta\varphi(x') = -f(\varphi(x'))$ and local elliptic estimates [25] imply that the same is true for the first order derivatives of φ. Differentiating the equation and iterating the estimates a finite number of times, one eventually obtains that all derivatives of φ up to order $K+m+5$ decay exponentially, proving the claim.

Finally, to verify hypothesis (ND) with $n=2$, we take

$$\omega_1(s) := \sqrt{(\lambda+s)|\mu_0|}, \quad \omega_2(s) := \sqrt{(\lambda+s)|\mu_0|+1},$$

$\omega(s) := (\omega_1(s), \omega_2(s))^T$, and compute the determinant of the 2×2 matrix $\left[\, \omega'(0) \ \ \omega(0) \,\right]$:

$$\begin{aligned}\det \left[\, \omega'(0) \ \ \omega(0) \,\right] &= \frac{|\mu_0|}{2}\left(\frac{\sqrt{\lambda|\mu_0|+1}}{\sqrt{\lambda|\mu_0|}} - \frac{\sqrt{\lambda|\mu_0|}}{\sqrt{\lambda|\mu_0|+1}} \right) \\ &= \frac{|\mu_0|}{2} \frac{1}{\sqrt{\lambda|\mu_0|(\lambda|\mu_0|+1)}} \neq 0.\end{aligned}$$

Hence, (ND) holds as well and we may now apply Theorem 2.3 with $n=2$.

Let $W \subset \mathbb{R}^2$ be as in the conclusion of Theorem 2.3. Thus for any $\bar{\omega} \in W$ there exist $s \in (-\delta, \delta)$ and a solution $v(x,y)$ of the equation

$$\Delta v + v_{yy} + a(x;s)v + f_1(x,v;s) = 0 \quad (x \in \mathbb{R}^N,\ y \in \mathbb{R}),$$

such that (1.5) holds with u replaced by v, and $v(x,y)$ is radially symmetric in x', even and 2π-periodic in x_N, and quasiperiodic in y with the frequency vector $\bar{\omega}$. By the definition of a and f_1, $\tilde{u} = \varphi^{\lambda+s} + v$ is a solution of

$$\Delta \tilde{u} + \tilde{u}_{yy} + (\lambda+s)f(\tilde{u}) = 0 \quad (x \in \mathbb{R}^N,\ y \in \mathbb{R}),$$

with the same properties as v. Using the rescaling $u(x,y) = \tilde{u}(x(\lambda+s)^{-1/2}, y(\lambda+s)^{-1/2})$ we obtain a solution of the original equation (1.1) which satisfies (1.5), and is radially symmetric in x', even and $2\pi(\lambda+s)$-periodic in x_N, and quasiperiodic in y with the frequency vector $(\lambda+s)\bar{\omega}$ (obviously, any such vector is nonresonant, just as $\bar{\omega}$). Since no two vectors in (the uncountable set) W are linearly dependent, the set of frequency vectors obtained this way is uncountable. So we have a family of solutions of (1.1) with the desired properties, we just need verify that they are all positive. This follows from (2.3). Indeed, let u be any of these solutions. Since it is quasiperiodic (in the sense of our definition), it is not periodic in y and in particular $u \not\equiv 0$. By the strong maximum principle, either $u > 0$ or u is negative somewhere. In the latter case, quasiperiodicity and (1.5) imply that u has a local negative minimum at some point. But at that point equation (1.1) cannot be satisfied when (2.3) holds. Thus $u > 0$.

The proof of Theorem 2.1 is now complete.

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