It is interesting to evaluate some of the approximations using a calculator. Thus for the integral \( \int_0^3 e^{x^2} \, dx \), the formula for \( M_6 \) is on p. 155, and \( T_6 \) and \( S_6 \) are on p. 156. Plugging everything into the calculator we find \( M_6 \approx 1,055.8 \), \( T_6 \approx 2,319.1 \), \( S_6 \approx 1,723.3 \).

A very accurate value of \( \int_0^3 e^{x^2} \, dx \) can be found using MATHEMATICA. Keeping only two decimal places, we find \( \int_0^3 e^{x^2} \, dx \approx 1,444.55 \).

Note that \( T_6 \) overshoots the most while \( M_6 \) is significantly less than the correct value. One can find an explanation for this by examining the graph of \( e^{x^2} \). We shall show an explanation for the Trapezoidal Rule which is the easiest to picture.
The point of this picture is on each of the intervals of the partition, the function $e^{x^2}$ grows much faster near the end of such an interval than at its beginning. This produces the effect that the area of the approximating trapezoid is a bit larger than the area under the corresponding part of the graph.

A magnified picture of this effect:

The trapezoid area is quite a bit larger than under the graph.

\[ \approx 55 \approx e^{2^2}, \text{ i.e., } e^{x^2} \text{ at } 2 \text{ equals } \approx 55, \]

but $e^{x^2}$ at 3 equals $\approx 8,103$.  

Graph of $e^{x^2}$ on $[0,3]$ and some of the approximating trapezoids.
Let's take a look at the errors. The errors $E_M$, $E_T$, $E_S$ are defined on pages 153, 154: Let's calculate these errors for $\int_0^3 e^x\,dx$ when $n = 6$:

$$E_M = \int_0^3 e^x\,dx - M_6 \approx 1,444.55 - 1,055.8 = 388.75$$

$$E_T = \int_0^3 e^x\,dx - T_6 \approx 1,444.55 - 2,319.1 = -874.55$$

$$E_S = \int_0^3 e^x\,dx - S_6 \approx 1,444.55 - 1,722.3 = -277.75$$

**Picture Explanation why the Midpoint Rule rectangles come up short:**

Simpson's Rule is typically more accurate than the other two rules.
Question: How large should we take \( n \) in order to guarantee that the Simpson's Rule approximation for \( \int_{0}^{\pi/2} \sin x \, dx \) is accurate to within 0.0001?

We need to choose \( n \) so that the error \( E_S \), in absolute value, is < 0.0001. From the Box on p. 514 in the Book we obtain

\[
|E_S| \leq \frac{K \cdot (b-a)^5}{180 \cdot h^4}
\]

where \( K \) is such that \( |f^{(4)}(x)| \leq K \) on \([a,b]\); \( f^{(4)}(x) \) means the fourth derivative of \( f(x) \) (= \( \sin x \) in our Example). So we have \( f''(x) = \cos x \), \( f'''(x) = -\sin x \), \( f''''(x) = -\cos x \), \( f^{(4)}(x) = \sin x \).
Now we use the important fact that $|\sin x| \leq 1$ for all $x$ (and likewise for $\cos x$). Actually, on the interval $[0, 1]$, $0 \leq \sin x \leq \sin 1 < 1$, but that is not a very large difference, so we will use $K = 1$.

Also $b - a = 1 - 0 = 1$, hence we obtain

$$|E_S| \leq \frac{1}{180n^4} = \frac{1}{180n^4}$$

Thus if we choose $n$ so that

$$\frac{1}{180n^4} < 0.0001$$

then we will obtain $|E_S| < 0.0001$.

Furthermore, note that $n$ must be even for using Simpson's Rule.

Multiply be $10^4$:

$$\frac{10^4}{180n^4} < 1 \implies \frac{10^4}{180} < n^4$$
So \( n > \sqrt[4]{\frac{10^4}{180}} \approx 2.73 \)

Thus \( n \) must be at least 3, and since \( n \) must be even, we can choose \( n = 4 \). (*

(We cannot choose \( n = 3 \).

Let's answer the Question at the top of p.166 with Simpson's Rule replaced by the Midpoint Rule:

The formula for the bound on \( |E_M| \)

is \( |E_M| \leq \frac{K(b-a)^3}{24n^2} \) (130x, p. 510)

where \( |f''(x)| \leq K \) on \([a, b]\)

As before \( [a, b] = [0, 1] \), hence \( b-a = 1-0 = 1 \), and \( f''(x) = (\sin x)'' = -\sin x \), hence \( K = 1 \).

Thus we can choose \( K = 1 \).
Thus \( |E_M| \leq \frac{1}{24n^2} \)

Hence \( |E_M| < 0.0001 \) if we choose \( n \)

so that \( \frac{1}{24n^2} < 0.0001 \)

Multiply by \( 10^4 \): \( \frac{10^4}{24n^2} < 1 \)

Multiply by \( n^2 \): \( \frac{10^4}{24} < n^2 \)

i.e. \( n > \sqrt{\frac{10^4}{24}} \approx 20.41 \)

Hence we can choose \( n = 21 \).
Let us return to the discussions of the approximations.

$M_6 \approx 1,055.8$, $T_6 \approx 2,319.1$, $S_6 \approx 1,722.3$

For the integral

$$\int_0^3 e^x \, dx \approx 1,444.55$$

which is an accurate value obtained by MATHEMATICA.

On p. 160 we obtained bounds for the errors $E_M$, $E_T$ and on p. 161 a bound for $E_S$. We had

$$|E_M| \leq \frac{171e^9}{4n^2}, \quad |E_T| \leq \frac{171e^9}{2n^2},$$

$$|E_S| \leq \frac{2,349e^9}{n^4}$$

Thus, with $n=6$ we obtain $|E_M| \leq 9,622.42$, but in fact, the actual error is much less than that, namely $E_M \approx 1,444.55 - 1,055.8$

$E_M \approx 388.75$. Similarly for $E_T$ and $E_S$. 
A comment to this effect is on p. 510, above Example 2: The Actual Error is often substantially smaller than what we obtain from the formulas for the error bounds:

\[ |E_T| \leq \frac{K(b-a)^3}{12n^2}, \quad |E_M| \leq \frac{K(b-a)^3}{24n^2} \]

\[ |E_S| \leq \frac{K(b-a)^5}{180n^4} \quad (K \text{ is different for } E_S \text{ from that for } E_T, E_M) \]

Another Comment. An important part of the information about the error bounds is that \( E_T \) and \( E_M \) approach zero at the rate \( \frac{1}{n^2} \) (i.e., some constant times \( \frac{1}{n^2} \)), whereas \( E_S \) approaches zero at the rate \( \frac{1}{n^4} \). Thus Simpson’s Rule is more accurate.
A comment on the Homework Question #29, p. 517:

How to pick the values of the function in the graph: At integer points, pick the values to be integers — that’s reasonable enough, looking at the graph.

At the midpoints pick the nearest multiple of \( \frac{1}{4} \). So for example, pick \( f(5.5) = 3.25 \).

By doing what is suggested here you will not get the same answers as listed in the Answers Section in the back of the Book.

HW: 7.5, 7.7, part of 7.8, through #37 (inclusive)
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