Sect. 8.1, 8.2.

The key formula for the length of the curve \( y = f(x), \ a \leq x \leq b, \) i.e., the length of the graph of \( f(x), \ a \leq x \leq b, \) is the formula in the large box on p. 539:

\[
L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx.
\]

Examples (Exercises, Exam Questions) involve calculating the length of the curve \( y = f(x), \ a \leq x \leq b, \) for various functions \( f. \) This requires strong integration skills as one should have learned from Ch. 7, as well as strong algebra skills to be able to simplify integrals of the form near the top of this page.

HW due 2/18: None

HW due 2/20: 7.8: 41, 49, 51, 63
8.1: 1, 3, 7, 9, 17, 23; 8.2: 3, 5, 7.
For Example: Find the length of the curve \( y = \frac{1}{2} x^2 \), \( 0 \leq x \leq 2 \).

We have \( \frac{dy}{dx} = x \), hence

\[
L = \int_{0}^{2} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_{0}^{2} \sqrt{1 + x^2} \, dx
\]

So we first calculate the antiderivative (See Also Top of p. 550 in the Book)

\[
\int \sqrt{1 + x^2} \, dx = \int \sec^2 \theta \, d\theta
\]

\[
\int \sqrt{1 + x^2} \, dx = \int \frac{1}{\sin^2 \theta} \, d\theta = \int \sec^2 \theta \, d\theta
\]

This is a somewhat complicated integral which we learned to do in the section 7.2 (Trigon. Integrals), and

\[
= \frac{1}{2} \left[ \sec \theta \tan \theta + \ln | \sec \theta + \tan \theta | \right] + C
\]

\[
= \frac{1}{2} \left[ x \sqrt{1 + x^2} + \ln | \sqrt{1 + x^2} + x | \right] + C
\]

The integral \( \int \sec^3 \theta \, d\theta \) is calculated on p. 510 of the Notes.
\[
\text{Hence } \int_0^2 \sqrt{1 + x^2} \, dx \\
= \frac{1}{2} \left[ x\sqrt{1 + x^2} + \ln |\sqrt{1 + x^2} + x| \right]_0^2 \\
= \frac{1}{2} \left[ 2\sqrt{5} + \ln |\sqrt{5} + 2| \right] \\
- \frac{1}{2} \left[ 0 + \ln 1 \right] \\
= \frac{1}{2} \left[ 2\sqrt{5} + \ln (\sqrt{5} + 2) \right] \\
= \frac{1}{2} \left[ 2\sqrt{5} + \ln (\sqrt{5} + 2) \right] \\
\]

An Easy Example, but one you should learn (as all Examples in the Book) is Example 1, p. 539 in the Book.

Example 2, p. 540, is similar to the Example started at the top of page 230 of the Notes, but requires substitution \( y = \frac{1}{2} \tan \theta \) (the variable of integration is \( y \)), not just \( y = \tan \theta \).
Example 4, p. 542: It does not require any special integration skills, but requires good algebra skills.

Let's work out a similar Example, namely Exercise #10, p. 543:

Find the exact length of the curve

\[ x = \frac{y^4}{8} + \frac{1}{4y^2} \quad 1 \leq y \leq 2. \]

\[
\frac{dx}{dy} = 4 \cdot \frac{y^3}{8} + \frac{1}{4} (-2) \frac{1}{y^3} = \]

\[
= \frac{1}{2} y^3 - \frac{1}{2y^3} = \frac{1}{2} \left( y^3 - \frac{1}{y^3} \right)
\]

We now calculate the quantity

\[ 1 + \left( \frac{dx}{dy} \right)^2 = 1 + \frac{1}{4} \left( y^3 - \frac{1}{y^3} \right)^2 = \]

\[
= 1 + \frac{1}{4} \left( y^6 - 2y^3 \cdot \frac{1}{y^3} + \frac{1}{y^6} \right) =
\]

\[
= 1 + \frac{1}{4} y^6 - \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{y^6} = \frac{1}{4} y^6 + \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{y^6}
\]

\[
= \frac{1}{4} \left( y^6 + 2 + \frac{1}{y^6} \right) = \frac{1}{4} \left( y^3 + \frac{1}{y^3} \right)^2
\]
Hence \( \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{\frac{1}{4} \left( y^3 + \frac{1}{y^3} \right)^2} \)

\[ = \frac{1}{2} \left( y^3 + \frac{1}{y^3} \right) \]

Hence the length of the curve in Exercise 10 is

\[ \int_{1}^{2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_{1}^{2} \frac{1}{2} \left( y^3 + \frac{1}{y^3} \right) \, dy \]

\[ = \left[ \frac{1}{8} y^4 + \frac{1}{2} \left( -\frac{1}{2} y^2 \right) \right]_{1}^{2} \]

\[ = \left( \frac{1}{8} \cdot 16 - \frac{1}{16} \right) - \left( \frac{1}{8} - \frac{1}{4} \right) = \]

\[ = 2 - \frac{1}{16} - \frac{1}{8} + \frac{1}{4} = \frac{33}{16} \]
The idea of simplifying the square root of \( \sqrt{1 + (f'(x))^2} \) in the manner shown in the preceding Example occurs in many problems on calculating the length of a curve. As mentioned already, Example 4 on p. 542 is like that.

Another frequent example is:

\[ y = \frac{1}{2} (e^x + e^{-x}) \]

hence \( \frac{dy}{dx} = \frac{1}{2} (e^x - e^{-x}) \),

and \( 1 + \left( \frac{dy}{dx} \right)^2 = 1 + \frac{1}{4} (e^{2x} - 2e^x e^{-x} + e^{-2x}) = 1 + \frac{1}{4} e^{2x} - \frac{1}{2} + \frac{1}{4} e^{-2x} = \frac{1}{4} (e^{2x} - 2e^x + e^{-2x}) = \frac{1}{4} (e^x + e^{-x})^2 \)

Hence \( \sqrt{\frac{1}{4} (e^x + e^{-x})^2} = \frac{1}{2} (e^x + e^{-x}) \).
Section 8.2 deals with the area of the surface obtained by rotating a graph/curve about the $x$- or $y$-axis. Thus if the curve $y = f(x)$, $a \leq x \leq b$ is rotated about the $x$-axis, the area of the surface obtained in this manner is

$$\int_{a}^{b} 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx$$

$= \pi \cdot \text{Box} \, [4]$, p. 547.

If the curve is given by $x = g(y)$, $c \leq y \leq d$, the area is given by

$$\int_{c}^{d} 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$= \pi \cdot \text{Box} \, [6]$, p. 547.
For rotation about the y-axis, the corresponding formulas are:

\[ y = f(x), \ a \leq x \leq b : \quad \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \]

\[ x = g(y), \ c \leq y \leq d : \quad \int_c^d 2\pi g(y) \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \]

When \( x = g(y) \), it becomes:

\[ y = f(x) \]

or, \( x = g(y) \)

Circle length: \( 2\pi x \)

Small portion of the graph:

\[ \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dx \]

OR:

\[ \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dy \]

Which is width of a thin strip/band cut out from the surface.
Two formulas for the area of a surface obtained by rotating a curve about the x-axis:

1. \( y = f(x) \Rightarrow \int_{a}^{b} 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx \)

2. \( x = g(y) \Rightarrow \int_{c}^{d} 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \)

A thin strip/band of width \( ds \):

\[ \sqrt{1 + (f'(x))^2} \, dx = 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx \]

or

\[ 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \]
Example. Calculate the area obtained by rotating the curve \( y = \frac{1}{2}x^2, 0 \leq x \leq 2 \), about the \( x \)-axis.

By the formula,

\[
S = \int_{a}^{b} 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx
\]

\[
S = \int_{0}^{2} 2\pi \cdot \frac{1}{2} x^2 \sqrt{1 + x^2} \, dx
\]

\[
= \pi \int_{0}^{2} x^2 \sqrt{1 + x^2} \, dx
\]

We calculate \( \int x^2 \sqrt{1 + x^2} \, dx \) first.

By subst. on p. 230, \( x = \tan \theta \)

We obtain \( \int \tan^2 \theta \sec^3 \theta \, d\theta \)

\[
= \int (\sec^2 \theta - 1) \sec^3 \theta \, d\theta = \int \sec^5 \theta \, d\theta - \int \sec^3 \theta \, d\theta
\]

on p. 52 in the Notes

on p. 51 in the Notes

We will not continue this any further.
Example. Find the area of the surface obtained by rotating the curve 
\( y = x^2 - \frac{1}{8} \ln x, \quad 1 \leq x \leq 2 \)
about the x-axis.

This curve is considered in Example 4 on p. 542 in the Book and it is calculated that

\[
\sqrt{1 + \left( \frac{dy}{dx} \right)^2} = 2x + \frac{1}{8x}
\]

Hence the area of the surface of this Example (top of this page) is given by

\[
\int_1^2 2\pi y \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

\[
= \int_1^2 2\pi \left( x^2 - \frac{1}{8} \ln x \right) \left( 2x + \frac{1}{8x} \right) \, dx
\]

\[
= 2\pi \int_1^2 \left( 2x^3 - \frac{1}{4} x \ln x + \frac{1}{8} x - \frac{1}{64} \frac{\ln x}{x} \right) \, dx
\]

and we know how to find all antiderivatives occurring in this integral. Finish on your own.
Section 8.3
Moments and Centers of Mass.
(We will not cover the Problems on Hydrostatic Pressure and Force.)

Simplest Case: Point Masses.

\[ \begin{array}{cccc}
  m_1 & m_2 & \cdots & m_k \\
  x_1 & x_2 & \cdots & x_k \\
  0 & x_1 & x_2 & x_k \\
\end{array} \]

Masses \( m_1, m_2, \ldots, m_k \) situated at points \( x_1, x_2, \ldots, x_k \) on the \( x \)-axis;
The center of mass \( \bar{x} \):

\[ \bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_k x_k}{m_1 + m_2 + \cdots + m_k} \]

If the masses are equal, say all equal to \( m \), we obtain

\[ \bar{x} = \frac{m x_1 + \cdots + m x_k}{m + m + \cdots + m} = \frac{m (x_1 + x_2 + \cdots + x_k)}{km} \]

i.e., \( \bar{x} = \frac{x_1 + x_2 + \cdots + x_k}{k} \).

i.e. the average of the coordinates.
\[ M = m_1x_1 + m_2x_2 + \ldots + m_kx_k = \sum_{i=1}^{k} m_i x_i \]

which is the numerator of the fraction in the formula for \( \bar{x} \), is called the moment of the system of masses about the origin.

For Example, \( \#22 \), bottom p. 561 in the Book 1: Find the moment \( M \) of the system about the origin and the center of mass \( \bar{x} \):

\[
\begin{array}{c}
m_1 = 12 & m_2 = 15 & m_3 = 20 \\
-3 & 0 & 2 & 8 \\
\end{array}
\]

\[ M = m_1x_1 + m_2x_2 + m_3x_3 = 12(-3) + 15(2) + 20(8) \]

\[ M = -36 + 30 + 160 = 154 \]

\[ \bar{x} = \frac{M}{m_1 + m_2 + m_3} = \frac{154}{12 + 15 + 20} = \frac{154}{47} \]

*total mass* \( \times \)
Systems of masses in the plane.

\[ m_1, m_2, \ldots, m_n \] located at \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\)

Then the moment of the system about the \(y\)-axis is

\[ M_y = \sum_{i=1}^{n} m_i x_i \] (page 555 in the Book)

The moment of the system about the \(x\)-axis is

\[ M_x = \sum_{i=1}^{n} m_i y_i \] (top p. 556 in the Book)

The total mass \( m = m_1 + m_2 + \ldots + m_n \)

Note the switch of \(x, y\), i.e.,

\( M_y \) is calculated from the \( x_i \),

i.e., the distances of the \( m_1, \ldots, m_n \) from the \(y\)-axis.

Similar switch occurs for \( M_x \).
The coordinates of the Center of Mass are

\[ \bar{x} = \frac{M_y}{m} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m} \]

\[ \bar{y} = \frac{M_x}{m} = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m} \]

i.e. To calculate \( \bar{x} \) we use \( x_1, x_2, \ldots, x_n \)

" \quad " \quad \( \bar{y} \) " \quad \( y_1, y_2, \ldots, y_n \)

**#24, p.561:** Find the Moments \( M_x \), \( M_y \) and the center of mass \((\bar{x}, \bar{y})\) if \( m_1 = 5, m_2 = 4, m_3 = 3, m_4 = 6 \), are situated at the points 

\( P_1 (-4, 2), P_2 (0, 5), P_3 (3, 2), P_4 (1, -2) \):

\[ M_x = m_1 \cdot y_1 + m_2 \cdot y_2 + m_3 \cdot y_3 + m_4 \cdot y_4 = \]

\[ = 5(2) + 4(5) + 3(2) + 6(2) = \]

\[ = 10 + 20 + 6 - 12 = 24 \]

\( M_y = m_1 \cdot x_1 + m_2 \cdot x_2 + m_3 \cdot x_3 + m_4 \cdot x_4 = \]

\[ = 5(-4) + 4(0) + 3(3) + 6(1) = -5 \]
The total mass \( m \),

\[ m = 5 + 4 + 3 + 6 = 18 \]

\[ \bar{x} = \frac{M_y}{m} = \frac{-5}{18} \]

\[ \bar{y} = \frac{M_x}{m} = \frac{24}{18} = \frac{4}{3} \]

Hence the center of mass is

\[ (\bar{x}, \bar{y}) = \left( -\frac{5}{18}, \frac{4}{3} \right) \]

\[ \star \]
The Moment of a horizontal thin wire of uniform density \( \rho \) about the y-axis:

\[
\begin{aligned}
\text{length } dx, \text{ mass}(\rho dx) \\
\end{aligned}
\]

So similarly to the formula
\[
M_y = m_1x_1 + m_2x_2 + \ldots + m_nx_n
\]

we have

\[
\int (\rho dx)x = \int \rho x \, dx
\]

which is a part of the "Sam" which is the integral
\[
M_y = \int_a^b (\rho dx)x = \int_a^b \rho x \, dx
\]

\[
= \rho \int_a^b x \, dx = \rho \left( \frac{1}{2} x^2 \right|_a^b = \frac{x}{2} (b^2 - a^2) = \frac{\rho(b-a)}{\text{mass}} \left[ \frac{1}{2} (a+b) \right]
\]