The Moment of a horizontal thin wire of uniform density \( \rho \) about the \( y \)-axis:

![Diagram showing the moment calculation]

- Length \( dx \), mass \((\rho dx)\)
- Density \( \rho \) is per unit length
- Approximated at the point \((x, y_0)\)

So similarly to the formula

\[
M_y = m_1x_1 + m_2x_2 + \cdots + m_nx_n
\]

we have

\[
(M_y) = (\rho dx)x
\]

which is a part of the "Sum"
which is the integral

\[
M_y = \int_a^b (\rho dx)x = \int_a^b \rho x dx
\]

\[
= \rho \int_a^b xdx = \rho \left( \frac{1}{2} x^2 \right) \bigg|_a^b
\]

\[
= \frac{1}{2} \rho (b^2 - a^2) = \rho \left( \frac{b-a}{2} \right) \left( \frac{1}{2} (a+b) \right) \frac{x}{mass}
\]
The moment of a horizontal thin wire of uniform density \( \rho \) about the \( x \)-axis:

\[
\text{mass} = \rho \, dx
\]

The density \( \rho \) is per unit length.

\( y_0 \) = the distance from \( x \)-axis

Thus, the wire can be thought as being built from small pieces of mass \( \rho \, dx \) very each of which has distance \( y_0 \) from the \( x \)-axis, hence it has moment \( y_0(\rho \, dx) \) about the \( x \)-axis.

The total moment of the entire wire about the \( x \)-axis is obtained by adding together all these moments \( y_0(\rho \, dx) \), over all \( x \) between \( a \) and \( b \), i.e., the integral:

\[
M_x = \int_a^b y_0(\rho \, dx) = \rho y_0 \, [b-a] = y_0 \rho [b-a] = \text{total mass} \text{distance from } x \text{-axis}
\]
Moments of a vertical thin wire of uniform density $\rho$ (per unit length):

\[ M_x = \int_c^d \rho y \, dy = \frac{1}{2} \rho (d^2 - c^2) = \frac{1}{2} \rho (c + d)(d - c) = \frac{1}{2} \rho (d - c) \left( \frac{1}{2} (c + d) \right) \]

\[ M_y = \int_c^d x_0 \rho \, dy = \rho x_0 (d - c) = x_0 \rho (d - c) \]

We proceed by analogy with the work for a horizontal wire:

We proceed by analogy with the work for a horizontal wire:
In general the moment of an extended mass (i.e., a line object about an axis (which can be any line (not just the x-axis or the y-axis) is calculated by dividing the object into very small pieces (i.e., of small size), multiplying the mass of each of these by its distance from the axis, then adding up the moments of these small pieces. Of course, the "adding up" usually means integrating.

A Principle for Calculating Moments.

The moment of an object about an Axis = (The Total Mass) times (the Distance of Center of Mass from the Axis)
Another Principle for Calculating the Moment About an Axis:
If an object is the union of a collection of disjoint pieces, then the moment of the object equals the sum of the moments of those pieces.

The Moments of a Thin Plate laid flat on the $x,y$-plane:
We also assume uniform density $\rho$ per unit area.

Shape of the Plate:

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center of mass of a rectangle is the center of the rectangle (uniform density assumed)
Thus the center of mass of the rectangle in the picture on preceding page has y-coord. \[ y = \frac{1}{2} (f(x) + g(x)), \]
and the x-coord. approx. equals \( x \), since \( dx \) is very small.

Moreover the area of the rectangle is \( (f(x) - g(x))dx \), hence its mass

\[ \rho (f(x) - g(x))dx \]

Thus the moment of the rectangle about \( x \)-axis equals

\[ \frac{1}{2} (f(x)+g(x)) \left[ \rho (f(x)-g(x))dx \right] \]

\[ \text{distance of its center of mass from } x \text{-axis} \]

\[ \text{mass of the rectangle} \]

Moment about \( x \)-axis, moment of the plate \( a \leq x \leq b, \ g(x) \leq y \leq f(x) \)

equals \( M_x = \int_a^b \frac{1}{2} (f(x)+g(x))\rho (f(x)-g(x)) \, dx \)
\[ \rho \int_{a}^{b} \frac{1}{2} \left[ (f(x))^2 - (g(x))^2 \right] dx \]

Thus the moment \( M_x \) of the plate of density 1 equals

\[ \int_{a}^{b} \frac{1}{2} \left[ (f(x))^2 - (g(x))^2 \right] dx \]

which is the called the moment of the region \( \{(x,y) : a \leq x \leq b, \ g(x) \leq y \leq f(x)\} \) about the \( x \)-axis.

The Moment about the \( y \)-axis:

For the rectangle in the picture on p. 249,

\[ \int \overbrace{x \rho(f(x)-g(x))}^{\text{mass}} \ dx \]

distance of center of mass from \( y \)-axis
The moment about the y-axis for the plate of density \( \rho \), and \( a \leq x \leq b, g(x) \leq y \leq f(x) \):

\[
M_y = \int_a^b x \rho (f(x) - g(x)) \, dx
\]

\[
= \rho \int_a^b x (f(x) - g(x)) \, dx
\]

If density \( \rho = 1 \), we obtain the moment \( M_y \) of the region \( \{(x, y) : a \leq x \leq b, g(x) \leq y \leq f(x)\} \):

\[
M_y = \int_a^b x (f(x) - g(x)) \, dx
\]

The \( x \) and \( y \) coordinates of the centroid of the region are obtained by dividing by the area:

\[
\bar{x} = \frac{M_y}{A}, \quad \bar{y} = \frac{M_x}{A}
\]

Compare this with the formulas near the top of p. 243.
We will do Exercise #32, p. 561 in the Book:
Find the centroid of the region bounded by the curves \( y = x^3 \), \( x + y = 2 \), \( y = 0 \).

\[
\begin{align*}
\text{We have to solve} & \quad y = x^3 \\
\text{Hence} & \quad x + x^3 = 2, \quad \text{thus} \quad x = 1, y = 1
\end{align*}
\]
Hence \( x + x^3 = 2 \), thus \( x = 1, y = 1 \) is a solution. Hence the drawing above shows the region. We divide it into the two regions \( R_1 \) and \( R_2 \). Thus the moments \( M_x \), \( M_y \) are obtained by adding the corresponding moments of \( R_1 \) and \( R_2 \).
First we calculate $M_x$ and $M_y$
for $R_1$. We have $0 \leq x \leq 1$,
$f(x) = x^3$, $g(x) = 0$. Hence

$$M_x = \int_0^1 \frac{1}{2} [f(x)]^2 \, dx = \int_0^1 \frac{1}{2} x^6 \, dx$$

$$= \left. \left( \frac{1}{2} \cdot \frac{1}{7} x^7 \right) \right|_0^1 = \frac{1}{14}$$

$$M_y = \int_0^1 x f(x) \, dx = \int_0^1 x \cdot x^3 \, dx$$

$$= \int_0^1 x^4 \, dx = \frac{1}{5} x^5 \bigg|_0^1 = \frac{1}{5}$$

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$M_x$ and $M_y$ for $R_2$:

$f(x) = 2 - x$ \hspace{1cm} (From $x + y = 2 \Rightarrow y = 2 - x = f(x)$)

$g(x) = 0$

$1 \leq x \leq 2$

$$M_x = \int_1^2 \frac{1}{2} [f(x)]^2 \, dx = \int_1^2 \frac{1}{2} (2-x)^2 \, dx$$

$$= \frac{1}{2} \cdot \frac{1}{3} (x-2)^3 \bigg|_1^2 = \frac{1}{6}$$
\[ M_y = \int_1^2 x f(x) \, dx = \int_1^2 x (2-x) \, dx \]

\[ = \int_1^2 (2x - x^2) \, dx = \left[ x^2 - \frac{1}{3} x^3 \right]_1^2 = \]

\[ = \left( 4 - \frac{8}{3} \right) - \left( 1 - \frac{1}{3} \right) = \]

\[ = 3 - \frac{8}{3} + \frac{1}{3} = 3 - \frac{7}{3} = \frac{2}{3} \]

Hence \((M_x \text{ for } R)\) equals \((M_x \text{ for } R_1) + (M_x \text{ for } R_2)\)

\[ = \frac{1}{14} + \frac{1}{6} = \frac{3+7}{42} = \frac{10}{42} = \frac{5}{21} \]

\((M_y \text{ for } R)\) equals \((M_y \text{ for } R_1) + (M_y \text{ for } R_2)\)

\[ = \frac{1}{5} + \frac{2}{3} = \frac{3+10}{15} = \frac{13}{15} \]
Now we need to calculate the area of $R$: equals area of $R_1$ plus Area of $R_2$,

\[(\text{Area of } R_1) = \int_0^1 x^3 \, dx = \frac{1}{4} x^4 \Big|_0^1 = \frac{1}{4}\]

\[(\text{Area of } R_2) \text{ obviously } = \frac{1}{2}\]

Hence (the area of $R$) $= \frac{3}{4}$

Finally

\[
\bar{x} = \frac{M_y}{A} = \frac{\frac{13}{15}}{\frac{3}{4}} = \frac{52}{45}
\]

\[
\bar{y} = \frac{M_x}{A} = \frac{\frac{5}{21}}{\frac{13}{15}} = \frac{5 \cdot 15}{21 \cdot 13} = \frac{5 \cdot 5}{7 \cdot 13} = \frac{25}{91}
\]

Hence the centroid of $R$ is

\[
(\bar{x}, \bar{y}) = \left( \frac{52}{45}, \frac{25}{91} \right)
\]