Section 10.2
Calculus with parametric curves.

First some formulas involving the Chain Rule as stated on p. 645:

We assume that a curve is defined parametrically: $x = f(t)$, $y = g(t)$ although the names of $f$, $g$ do not enter the formulas below:

Chain Rule: \[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}
\]

Hence \[
\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \left(\frac{dx}{dt}\right)^{-1}
\]

Example. Consider the curve defined parametrically by $x = e^t - 1$, $y = e^{2t}$ as on p. 305 of the Notes.

Then \[
\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = 2e^{2t}
\]

Hence \[
\frac{dy}{dx} = \frac{2e^{2t}}{e^t} = 2e^t,
\]

where \[
\frac{dy}{dx}
\]

is calculated using only differentiation with respect to $t$, not $w/r to x$. 
One could also first eliminate $t$ as on p. 305, obtaining $y = (x+1)^2$ and thus
\[ \frac{dy}{dx} = 2(x+1) \]
Since $x = e^t - 1$, substituting into we obtain
\[ \frac{dy}{dx} = 2(e^t - 1 + 1) = 2e^t \text{ as before.} \]

One can also obtain \( \frac{d^2y}{dx^2} \) by differentiating with respect to $t$ only:

Let’s first restate the Chain Rule again: For any $w$ (differentiable with resp. to $x$, and $x$ diff. with resp. we have
\[ \frac{dw}{dt} = \frac{dw}{dx} \frac{dx}{dt} \]
Thus if we set \( w = \frac{dy}{dx} \),

we obtain \( \frac{dw}{dt} = \frac{dw}{dx} \frac{dx}{dt} \),

hence \( \frac{dw}{dx} = \frac{\frac{dw}{dt}}{\frac{dx}{dt}} \).

But \( \frac{\frac{dw}{dx}}{\frac{dx}{dx}} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2} \).

Also \( \frac{dw}{dt} = \frac{d}{dt} \left( \frac{dy}{dx} \right) \), \( \frac{dw}{dx} = \frac{d}{dx} (w) \).

Hence \( \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left( \frac{dy}{dx} \right)}{\frac{dx}{dt}} \).

And, \( \left\{ \frac{dy}{dx} = \frac{dy}{dt} \frac{dx}{dt} \right\} \).

Hence all differentiations in the box above, on the right hand side, can be done by differentiating w.r.t to \( x \) only.
Thus returning to the example on p. 312, \( x = e^t - 1, \ y = e^{2t} \), we found \( \frac{dy}{dx} = 2e^t, \ \frac{dx}{dt} = e^t \), hence

\[
\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( 2e^t \right) = \frac{d}{dt} \left( 2e^t \right) = \frac{2e^t}{e^t} = 2;
\]

On the other hand, we found by eliminating \( t \) that

\[
\frac{dy}{dx} = 2(x+1) \quad \text{(top p. 313)}
\]

Hence differentiating directly with respect to \( x \), we obtain

\[
\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} \left( 2(x+1) \right) = 2
\]

which agrees with the result above.
More Complicated Example.

We use Example 2, p. 646 in the Book. The parameter is called $\Theta$ instead of $t$: $x = r(\Theta - \sin \Theta)$, $y = r(1 - \cos \Theta)$

Eliminating $\Theta$ is not a handy thing to do, hence calculating $\frac{dy}{dx}$ "directly" is not a good idea.

\[
\frac{dy}{dx} = \frac{dy}{d\Theta} \cdot \frac{d\Theta}{dx} = \frac{r \sin \Theta}{r - r \cos \Theta} = \frac{\sin \Theta}{1 - \cos \Theta}
\]

\[
\frac{d^2 y}{dx^2} = \frac{d}{d\Theta} \left( \frac{dy}{dx} \right) = \frac{d}{d\Theta} \left( \frac{\sin \Theta}{1 - \cos \Theta} \right)
\]

First calculate $\frac{d}{d\Theta} \left( \frac{\sin \Theta}{1 - \cos \Theta} \right) =$

\[
= \frac{\cos \Theta (1 - \cos \Theta) - \sin \Theta \sin \Theta}{(1 - \cos \Theta)^2} = \frac{\cos \Theta - \cos^2 \Theta - \sin^2 \Theta}{(1 - \cos \Theta)^2} = \frac{\cos \Theta - 1}{(1 - \cos \Theta)^2}
\]

\[
= \frac{1}{\cos \Theta - 1}
\]
Hence \[
\frac{d^2 y}{dx^2} = \frac{d}{d\theta} \left( \frac{\sin \theta}{1 - \cos \theta} \right) = \frac{1}{r - r\cos \theta} = \frac{1}{r(\cos \theta - 1)(1 - \cos \theta)} = \frac{1}{r(1 - \cos \theta)^2}
\]

Since this quantity is < 0

(for all values of \( \Theta \) which are not an integer or multiple of \( 2\pi \)),

it follows that the curve is concave down (when viewed as the graph of a function of \( x \)).
Example. Find the equation of the tangent line to the curve at the given point:
(a) \( x = e^t - 1, \ y = e^t \), point \((0,1)\)
(b) \( x = \theta - \sin \theta, \ y = 1 - \cos \theta \)

at the point corresponding to \( \theta = \frac{\pi}{4} \)

Solution:
(a) We have \( \frac{dy}{dx} = 2e^t \) \((p.312)\)
Moreover, \( x = 0 \), i.e. \( e^t - 1 = 0 \), i.e. \( e^t = 1 \), hence \( t = 0 \).
Thus when \( x = 0 \), we obtain \( \frac{dy}{dx} = 2e^t \bigg|_{t=0} \), i.e. \( \frac{dy}{dx} = 2e^0 = 2 \).

Hence the tangent line passes through \((0,1)\) and has slope = 2:
\( y - 1 = 2(x - 0) \), i.e.
\( y = 2x + 1 \)
(b) \( x = \theta - \sin \theta, \quad y = 1 - \cos \theta \),

depoint corresponding to \( \theta = \frac{\pi}{4} \):

\[ x = \frac{\pi}{4} - \sin \frac{\pi}{4} = \frac{\pi}{4} - \frac{\sqrt{2}}{2}, \]

\[ y = 1 - \cos \frac{\pi}{4} = 1 - \frac{\sqrt{2}}{2}. \]

\[
\frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta} = \frac{\sin \frac{\pi}{4}}{1 - \cos \frac{\pi}{4}} = \]

\[
= \frac{\sqrt{2}}{2} \frac{\sqrt{2}}{2 - \sqrt{2}} = \frac{\sqrt{2}}{2 - \sqrt{2}} = \frac{\sqrt{2}(2+\sqrt{2})}{2 - \sqrt{2}(2+\sqrt{2})} = \frac{\sqrt{2}(2+\sqrt{2})}{2} = \]

\[
= \frac{2\sqrt{2} + 2}{2} = \sqrt{2} + 1.
\]

\[ y - (1 - \frac{\sqrt{2}}{2}) = (\sqrt{2} + 1)(x - (\frac{\pi}{4} - \frac{\sqrt{2}}{2})). \]

*Not simplify further.*
If \( x = f(t), \ y = g(t) \) are the \((x, y)\)-coord. of a particle at time \( t \), then the distance traveled by the particle between \( t = \alpha \) and \( t = \beta \), i.e. \( \alpha \leq t \leq \beta \), is given by

\[
L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

This integral also equals the length of the curve defined by

\[
x = f(t), \ y = g(t), \quad \alpha \leq t \leq \beta
\]

provided the curve is traced out only once as \( t \) increases.

Example. Find the length of the curve defined by \( x = r(\theta - \sin \theta), \ y = r(1 - \cos \theta), \ \alpha \leq \theta \leq \beta \).

Solution. We have \( \frac{dx}{dt} = r(1 - \cos \theta) \), hence \( \frac{dx}{dt} \geq 0 \) and \( = 0 \) only when \( \theta \) is an integer multiple of \( 2\pi \). Hence \( x \) is strictly increasing as \( \theta \) increases, which means that
every point on the curve is visited only once. Thus we can use the formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

We have

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 =$$

$$\left(\frac{\pi}{2} (1 - \cos \theta)\right)^2 + \left(\frac{r \sin \theta}{2}\right)^2 =$$

$$\pi^2 \left(1 - 2\cos \theta + \cos^2 \theta\right) + r^2 \sin^2 \theta =$$

$$\pi^2 \left(1 - 2\cos \theta + \cos^2 \theta + \sin^2 \theta\right) =$$

$$\pi^2 \left(1 - 2\cos \theta + 1\right) = 4r^2 \left(\frac{1}{2} (1 - \cos \theta)\right) =$$

$$4r^2 \sin^2 \left(\frac{\theta}{2}\right)$$

Hence

$$L = \int_a^b \sqrt{4r^2 \sin^2 \left(\frac{\theta}{2}\right)} \, d\theta =$$

$$= \int_a^b 2r \left| \sin \left(\frac{\theta}{2}\right) \right| \, d\theta$$

Thus, if, for example, $0 \leq \theta \leq 2\pi$, then $0 \leq \frac{\theta}{2} \leq \pi$, hence $\sin \frac{\theta}{2} > 0$
\[ \left| \sin \frac{\theta}{2} \right| = \sin \frac{\theta}{2} , \text{ hence} \]

\[
\int_0^{2\pi} 2r \left| \sin \frac{\theta}{2} \right| d\theta = \int_0^{2\pi} 2r \sin \frac{\theta}{2} d\theta
\]

\[
= \left[ 2r \cdot 2 \left( -\cos \frac{\theta}{2} \right) \right]_0^{2\pi} =
\]

\[
4r \left( -\cos \pi - (-\cos 0) \right) =
\]

\[
4r \left( (-1) - (-1) \right) = 8r
\]

On the other hand, if \(2\pi \leq \theta \leq 3\pi\),
then \(\pi \leq \frac{\theta}{2} \leq \frac{3}{2}\pi\), hence \(\sin \left( \frac{\theta}{2} \right) \leq 0\)
and thus \(\left| \sin \left( \frac{\theta}{2} \right) \right| = -\sin \frac{\theta}{2}\), and
thus

\[
L = \int_{2\pi}^{3\pi} 2r \left| \sin \left( \frac{\theta}{2} \right) \right| d\theta =
\]

\[
= \int_{2\pi}^{3\pi} 2r \left( -\sin \frac{\theta}{2} \right) d\theta = \left[ 4r \cos \frac{\theta}{2} \right]_{2\pi}^{3\pi}
\]

\[
= 4r \left( \cos \frac{3\pi}{2} - \cos \pi \right) = 4r \left( 0 - (-1) \right) = 4r
\]
The Area of the Surface Obtained by Rotating a parametrically defined curve about the x-axis or about the y-axis is:

This is similar to what we did in the section 8.2, pages 235-239 in the Notes.

For example, on p. 235, if the surface is obtained by rotating the curve \( y = f(x) \), \( a \leq x \leq b \), about the x-axis, then the area equals

\[
A = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx
\]

or

\[
\int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx
\]

If the curve is defined parametrically, where \( x, y \) are functions of \( t \), \( a \leq t \leq b \), then

\[
A = \int_a^b 2\pi y \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt
\]
For Example, #62, p. 652 in the Book:

Find the exact area of the surface obtained by rotating the curve

\[ x = 3t - t^3, \quad y = 3t^2, \quad 0 \leq t \leq 1 \]

about the x-axis:

\[
A = \int_0^1 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

First \[ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3-3t^2)^2 + (6t)^2 \]

\[
= 9 - 18t^2 + 9t^4 + 36t^2 = 9t^4 + 18t^2 + 9 = 9(t^4 + 2t^2 + 1)
\]

\[
= 9(t^2 + 1)^2,
\]

Hence \[ A = \int_0^1 2\pi y \sqrt{9(t^2 + 1)^2} \, dt \]

\[
= \int_0^1 2\pi y \cdot 3(t^2 + 1) \, dt = \int_0^1 6\pi \cdot 3t^2(t^2 + 1) \, dt = 18\pi \left(\frac{t^5}{5} + \frac{t^3}{3}\right)
\]

\[
= 18\pi \left(\frac{1}{5} + \frac{1}{3}\right) = 18\pi \cdot \frac{8}{15} = \frac{48}{5} \pi
\]

\[ \times \]
Rotating about the y-axis works similarly. The Area is given by

\[
\int_{a}^{b} 2\pi x \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, dt,
\]

where \( x \) is substituted by its expression in terms of the parameter \( t \).

For example for the surface from the Exercise #62, p. 652 in the Book (which we just did for rotation about the y-axis), the integral is

\[
\int_{0}^{1} 2\pi \left( 3t - t^3 \right) \cdot \sqrt{(3(1^2+1)} \, dt \times \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}
\]
Section 10.3, Polar Coordinates.

Every point in the plane can be assigned exactly one pair of \(x, y\)-coordinates but many different pairs of polar coordinates. Firstly, every point \(P(x, y)\) which is not the origin \((0,0)\), can be assigned exactly one pair of "basic" polar coordinates \((r, \theta)\) which are obtained as follows:

![Polar Coordinates Diagram]

The first coordinate, denoted by \(r\), is the distance from the origin. The 2nd coordinate, denoted by \(\theta\), is the angle between the segment \(PO\) joining
$P$ to the origin and the positive direction of the $x$-axis, measured from the positive direction of the $x$-axis to the segment $PO$, in counterclockwise direction.

Furthermore, the point $P$ with "basic" polar coordinates $(r, \theta)$ is also assigned additional polar coordinates $(r, \theta + 2n\pi)$ where $n$ is any integer — positive, negative (or zero) — we then obtain the original coordinates $(r, \theta)$. 
The "basic" polar coordinates of a point $P \neq O$ are then those polar coordinates $(r, \theta)$ where $r > 0$ and $0 \leq \theta < 2\pi$. Thus we will not use the word basic officially, instead we can say "that pair of polar coordinates where $r > 0$ and $0 \leq \theta < 2\pi".

**Negative $\theta$:** Consider e.g. the point $P$ with polar coordinates $(2, -\frac{\pi}{4})$. Then $(2, -\frac{\pi}{4} + 2n\pi)$ are also polar coordinates of the same point, where $n$ is any integer. In particular $(2, -\frac{\pi}{4} + 2\pi) = (2, \frac{7}{4}\pi)$ is a pair of polar coordinates of this point.
Thus a negative angle means to measure the angle of its absolute value in counterclockwise direction.

Polar Coordinates of the origin:

Any pair \((0, \theta)\), i.e. the first coordinate \(= 0\) (distance from the origin), but the 2nd coordinate can be any angle whatsoever.

Negative \(r\): We agree that the point \(P\) with coordinates \((r, \theta)\) will also be assigned coordinates \((-r, \theta + \pi)\), and thus also \((-r, \theta + (2n+1)\pi)\)
where \( n \) is any integer.

Thus if a point \( P \neq 0 \) has polar coordinates \((r, \theta)\), then all pairs of its polar coordinates are

\[(r, \theta + 2n\pi) \text{ and } (-r, \theta + (2n+1)\pi),\]

where \( n \) is any integer.

A picture for negative \( r \).