The Main Definitions and Theorems.

The URT for $S$-wffs, (or the $S$-URT, or the URT for $\tilde{S}$).

Let $\alpha$ be an $S$-wff, i.e. $\alpha$ belong to $\tilde{S}$ (see p. 65 in the Notes). Then exactly one of the six possibilities happens:

1. There exists a unique (i.e. exactly one) natural number $n$ such that $\alpha$ is $A_n$, and $A_n$ belongs to $S$.

2. There exists a unique (i.e. exactly one) $S$-wff $\beta$, i.e. $\beta$ belonging to $\tilde{S}$, such that $\alpha$ is $(\neg \beta)$. Moreover $\beta$ is also the unique wff (ordinary wff) such that $\alpha$ is $(\neg \beta)$.

3. There exists a unique (i.e. exactly one) $S$-wff $\beta$, i.e. $\beta$ belonging to $\tilde{S}$, and a unique (i.e. exactly one) $S$-wff $\gamma$, i.e. $\gamma$ belonging to $\tilde{S}$, such that $\alpha$ is $(\beta \land \gamma)$. Moreover, $\beta$ and $\gamma$ are also the unique (ordinary) wffs such that $\alpha$ is $(\beta \land \gamma)$.

4)-(6) are similar to (3), with $\alpha$ being $(\beta \lor \gamma)$, $(\beta \rightarrow \gamma)$, $(\beta \leftrightarrow \gamma)$ respectively.

Comment. The "ordinary/basic" URT is the particular case of the URT for $S$-wffs when $S' = \{A_1, A_2, \ldots, A_n, \ldots\}$, i.e. $\forall S'$ is the set of all sentence symbols.

$p. 54.1, 55$ in the handwritten notes, or $p. 34, 35$ in typed Notes numbered pages 19-37.

So to get the "basic" URT from the typed up statement at the upper part of this page, we need to set $S' = \{A_1, A_2, \ldots, A_n, \ldots\}$, and change $S$-wff (and/or $\alpha$ belongs to $\tilde{S}$) to just "ordinary" wff.
The $S_1, S_2$-UR Agreement Theorem.

Suppose that $\alpha$ is both an $S_1$-wff, as well as an $S_2$-wff. Then $\alpha$ falls under exactly the same unique case, of the six cases of the $S_1$-URT as for the $S_2$-URT. Moreover it is the same unique case, that $\alpha$ falls under, for the "basic" URT. Furthermore, for each of the cases (1)-(6), the following holds:

Case (1). If $\alpha$ falls under this unique case for all three URTs ($S_1, S_2, \text{basic}$), then there is a unique natural number $n$ such that the sentence symbol $A_n$ belongs to $S_1$ as well as $S_2$, and $\alpha$ is $A_n$.

Case (2). If $\alpha$ falls under this unique Case for all three URTs, then there is a unique wff $\beta$ such that $\beta$ is also an $S_1$-wff, as well as an $S_2$-wff, and $\alpha$ is $\neg \beta$. Thus $\alpha$ is expressed as $\neg \beta$ when we view $\alpha$ as an $S_1$-wff, as well as when we view $\alpha$ as $S_2$-wff, as well as when we view $\alpha$ as an ordinary wff.

Case (3). When $\alpha$ falls under this unique case for all three URTs, then there exists exactly one (unique) wff $\beta$, and exactly one (unique) wff $\gamma$ such that $\alpha$ is $\beta \wedge \gamma$. Moreover, $\beta$ and $\gamma$ are also the unique $S_1$-wffs, as well as the unique $S_2$-wffs such that $\alpha$ is $\beta \wedge \gamma$.

Cases (4)-(6), dealing with "$\lor", \ "\rightarrow", \ "\leftrightarrow"$ are similar to Case (3).

Another key Theorem/Result is what we have been calling the **FACT** which is typed up in various handouts (also emailed) and is originally stated on p. 66 in handwritten notes:

**The FACT.**

Let $S'$ be a set of sentence symbols, and $\alpha$ be an expression of LSL. Then the following are equivalent:

(a) $\alpha$ is an $S'$-wff, i.e., $\alpha$ belongs to $S'$;

(b) $\alpha$ is a wff (i.e., an ordinary wff) and all sentence symbols that occur in $\alpha$ belong to $S'$.

See the last page of this attachment.
The Truth Assignment Theorem (Theorem 12A in the Book)

For any truth assignment \( \nu \) for a set \( S \) (of sentence symbols) there is a UNIQUE function \( \bar{\nu} : \bar{S} \to \{F, T\} \) satisfying the conditions 0. - 5.

0. For every sentence symbol \( A_i \) in \( S \), the value \( \bar{\nu}(A_i) \) is equal to the given value \( \nu(A_i) \), i.e. \( \bar{\nu}(A_i) = \nu(A_i) \).

1. \( \bar{\nu}(\neg \alpha) = \begin{cases} T & \text{if } \bar{\nu}(\alpha) = F \\ F & \text{if } \bar{\nu}(\alpha) = T \end{cases} \)

which has to hold for every wff \( \alpha \) belonging to \( \bar{S} \), i.e. for every \( S \)-wff.

2. \( \bar{\nu}((\alpha \land \beta)) = \begin{cases} T & \text{if } \bar{\nu}(\alpha) = T \text{ and } \bar{\nu}(\beta) = T \\ F & \text{if } \bar{\nu}(\alpha) = T \text{ and } \bar{\nu}(\beta) = F \\ \text{or } \bar{\nu}(\alpha) = F \text{ and } \bar{\nu}(\beta) = T \\ \text{or } \bar{\nu}(\alpha) = F \text{ and } \bar{\nu}(\beta) = F \end{cases} \)

i.e. \( \bar{\nu}((\alpha \land \beta)) = T \) if both \( \bar{\nu}(\alpha) = T \) and \( \bar{\nu}(\beta) = T \), and \( \bar{\nu}((\alpha \land \beta)) = F \) if either one of \( \bar{\nu}(\alpha), \bar{\nu}(\beta) = F \), or both \( \bar{\nu}(\alpha), \bar{\nu}(\beta) = F \).

The condition 2., as well as the conditions 3., 4., 5. below, have to hold for every pair of wffs \( \alpha, \beta \) in \( \bar{S} \), i.e. for every pair \( \alpha, \beta \) of \( S \)-wffs.

3. \( \bar{\nu}((\alpha \lor \beta)) = \begin{cases} T & \text{if } \bar{\nu}(\alpha) = T \text{ or } \bar{\nu}(\beta) = T \\ F & \text{if } \bar{\nu}(\alpha) = F \text{ and } \bar{\nu}(\beta) = F \end{cases} \)

4. \( \bar{\nu}((\alpha \rightarrow \beta)) = \begin{cases} T & \text{if } \bar{\nu}(\alpha) = F \text{ or } \bar{\nu}(\beta) = T \\ F & \text{if } \bar{\nu}(\alpha) = T \text{ and } \bar{\nu}(\beta) = F \end{cases} \)

5. \( \bar{\nu}((\alpha \leftrightarrow \beta)) = \begin{cases} T & \text{if } \bar{\nu}(\alpha) = \bar{\nu}(\beta) = T, \text{ or } \bar{\nu}(\alpha) = \bar{\nu}(\beta) = F \\ F & \text{if } \bar{\nu}(\alpha) = T \text{ and } \bar{\nu}(\beta) = F, \text{ or } \bar{\nu}(\alpha) = F \text{ and } \bar{\nu}(\beta) = T. \end{cases} \)
The following theorem is the Exercise 6 at the bottom of p.27 and top of p.28 in the Book.

**The Truth Assignments Agreement Theorem**

Let $S_1$ and $S_2$ be sets of sentence symbols and $\nu_1 : S_1 \to \{T, F\}$, $\nu_2 : S_2 \to \{T, F\}$ be truth assignments. Let

$$S = \{A_i \in S_1 \cap S_2 : \nu_1(A_i) = \nu_2(A_i)\}$$

Then for every $S$-wff $\alpha$ (i.e. $\alpha$ belonging to $\tilde{S}$), we have

$$\tilde{\nu}_1(\alpha) = \tilde{\nu}_2(\alpha).$$

**Proof.**

First some comments: $\tilde{\nu}_1 : \tilde{S}_1 \to \{T, F\}$, $\tilde{\nu}_2 : \tilde{S}_2 \to \{T, F\}$ are as obtained in Theorem 12A.

If $\alpha$ is in $\tilde{S}$, i.e. if $\alpha$ is an $S$-wff, then $S(\alpha) \subset S \subset S_1 \cap S_2 \subset S_1'$ and $\alpha \in \tilde{S}_2$. Thus $\tilde{\nu}_1(\alpha)$, $\tilde{\nu}_2(\alpha)$ are both defined. If thus remains to be proved that $\tilde{\nu}_1(\alpha) = \tilde{\nu}_2(\alpha)$ for every such $\alpha$ in $\tilde{S}$. We will prove this by induction on $\ell(\alpha)$, as we have done with other theorems, notably the FACT.

**Base Case.**

$\ell(\alpha) = 1$

$\alpha$ is an $S$-wff, so $\alpha$ is a wff, so $\ell(\alpha) = 1$ means $\alpha$ is a sentence symbol, i.e. $\alpha$ is $A_i$ for a unique natural number $i$. But $\alpha$ is an $S$-wff, so $A_i$ belongs to $S$. Thus by the Definition of $S$, we obtain $\nu_1(A_i) = \nu_2(A_i)$.

By the condition 0. of theorem 12A, we obtain $\tilde{\nu}_1(A_i) = \nu_1(A_i)$, $\tilde{\nu}_2(A_i) = \nu_2(A_i)$, hence $\tilde{\nu}(A_i) = \tilde{\nu}_2(A_i)$. But $A_i$ is $\alpha$, hence $\tilde{\nu}_1(\alpha) = \tilde{\nu}_2(\alpha)$.

Recall that $S'(\alpha)$ is the set of those sentence symbols that occur in $\alpha$. Thus if $S'(\alpha) \subset S'$, which in turn is contained in both $S'_1$, $S'_2$, then all sentence symbols occurring in $\alpha$ belong to both $S'_1$ as well as $S'_2$. Thus $\alpha$ is both an $S'_1$-wff, as well as an $S'_2$-wff. I.e., also $\alpha \in \tilde{S}'_1$, $\alpha \in \tilde{S}'_2$. 


Induction Hyp

Suppose $\bar{\nu}_1(\beta) = \bar{\nu}_2(\beta)$, $\bar{\nu}_1(\gamma) = \bar{\nu}_2(\gamma)$, etc., for all $\beta, \gamma$, etc., belonging to $S$ such that $\ell(\beta), \ell(\gamma) \leq n$.

Suppose $\ell(\alpha) = n + 1 \geq 2$ (since $\ell(\alpha) = 1$ is done in the base case). $\alpha$ is an $S$-wff, hence $\alpha$ is both an $S_1$-wff, as well as $S_2$-wff (see previous page). Thus by the $\{S_1, S_2, \text{URT (pages 91, 92)}, \text{one of the Cases (2)-(6) must occur. Suppose Case (2) occurs. Thus (see near top of page 92) there exists a unique } \beta \text{ which is both an } S_1\text{-wff, as well as an } S_2\text{-wff, such that } \alpha = (\neg \beta). \text{ But } \beta \text{ is also an } S\text{-wff, hence } \beta \text{ is an } S\text{-wff, and } L(\beta) < n. \text{ Thus by the Induction Hyp, we obtain } \bar{\nu}_1(\beta) = \bar{\nu}_2(\beta). \text{ But then } \bar{\nu}_1((\neg \beta) = \bar{\nu}_2((\neg \beta)) \text{ since } \bar{\nu}_1((\neg \beta)) = \{ T \text{ if } \bar{\nu}_1(\beta) = F \text{, likewise } \bar{\nu}_2((\neg \beta)) = \{ T \text{ if } \bar{\nu}_2(\beta) = F \text{, } F \text{ if } \bar{\nu}_1(\beta) = T \text{, F if } \bar{\nu}_2(\beta) = T \}

\text{But } \bar{\nu}_1(\beta) = \bar{\nu}_2(\beta) \text{ by the Induction Hyp. Thus if } \bar{\nu}_1(\beta) = \bar{\nu}_2(\beta) = F, \text{ then } \bar{\nu}_1((\neg \beta)) = \bar{\nu}_2((\neg \beta)) = T, \text{ i.e., } \bar{\nu}_1(\alpha) = \bar{\nu}_2(\alpha) = T.

And if } \bar{\nu}_1(\beta) = \bar{\nu}_2(\beta) = T, \text{ then } \bar{\nu}_1((\neg \beta)) = \bar{\nu}_2((\neg \beta)) = F, \text{ i.e., } \bar{\nu}_1(\alpha) = \bar{\nu}_2(\alpha) = F.

But either } \bar{\nu}_1(\beta) = \bar{\nu}_2(\beta) = T, \text{ or } \bar{\nu}_1(\beta) = \bar{\nu}_2(\beta) = F. \text{ Thus since in each case } \bar{\nu}_1(\alpha) = \bar{\nu}_2(\alpha), \text{ the Method of Proof by Cases allows us to conclude that } \bar{\nu}_1(\alpha) = \bar{\nu}_2(\alpha). \text{ Thus the Case (2) of URT (i.e., negation) is completed. Case (3) (i.e., } \land \text{ ) of the } S_1, S_2 - \text{URT Agreement Thm. is also completed.}

Case (4) (i.e., } \lor \text{)}

Case (5) (i.e., } \rightarrow \text{)}

Case (6) (i.e., } \leftrightarrow \text{)}

Any of these cases, including (1), (2), (3), can be on the Exam, March 20,
We now define a set of wffs $\bar{S}$, i.e., $\bar{S}$-wffs, as at the bottom third of p. 10 in the Book, to consist of those wffs that can be obtained in the manner (a), (b), (c) at the bottom of p. 16 in the Book, but with the parts (a) (b) and (c) altered as follows:

(a): Every sentence symbol in $\bar{S}$ is an $\bar{S}$-wff.

(b): If $\alpha$ and $\beta$ are $\bar{S}$-wffs, then $\neg \alpha$, $(\alpha \land \beta)$, $(\alpha \lor \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ are $\bar{S}$-wffs.

(c): No expression is an $\bar{S}$-wff unless it is compelled to be one by (a) and (b).

We then define $\bar{S}$ to consist of all $\bar{S}$-wffs.

The bottom-up process of building up $\bar{S}$-wffs, i.e., wffs in $\bar{S}$ from sentence symbols in $\bar{S}$.

So this is the same as (a), (b), (c) bottom of p. 16 in Book, but adjusted so only sentence symbols in $\bar{S}$ are allowed.