INDUCTIVELY FACTORED SIGNED-GRAPHIC ARRANGEMENTS OF HYPERPLANES

G. DAVID BAILEY
LENOIR-RHYNE COLLEGE

Abstract. In 1994, Edelman and Reiner characterized free and supersolvable hyperplane arrangements in the restricted interval $[A_{n-1}, B_n]$. In this paper, we give a characterization of inductively factored arrangements in this interval, and show that the same characterization also describes factored arrangements in this interval. These results use the compact notation of signed graphs introduced by Zaslavsky.

1. Introduction

The topology of the complement of an arrangement of hyperplanes started to receive a significant amount of attention with the work of Arnold [Ar], and of Fox, Fadell and Neuwirth [FaN], [FoN], on the *braid arrangements* A_n in the 1960's. A portion of the work since then (by Saito [Sa], Terao [Te1-2], Stanley [St1-2], Falk and Randell [FR], Jambu and Paris [JP]) has been aimed at identifying properties of the braid arrangement and other related arrangements which lead to nice enumerative, topological and algebraic consequences. Specifically, much work has been done to determine conditions under which the *characteristic polynomial* of such an arrangement factors completely with integer roots.

Our goal is to describe such conditions among certain members of the restricted class of *signed-graphic* arrangements, which are the subarrangements of the reflecting hyperplanes for the classical reflection group B_n. This class contains all the arrangements associated with the classical reflection groups A_n, $B_n (= C_n)$, D_n. One nice property of these arrangements is that one can analyze and perform computations on such arrangements using the compact notation of *signed graphs*, a well behaved generalization of ordinary undirected graphs introduced by Harary [Ha]. Zaslavsky was the first to apply the theory of signed graphs to hyperplane arrangements [Za1].

The main result of this paper is part b) of Theorem 3.3, which gives a simple characterization of factored and inductively factored arrangements in the interval $[A_{n-1}, B_n]$. Theorem 3.3 combines this characterization with previous results by Edelman and Reiner [ER] to obtain a complete characterization for supersolvable, factored, inductively factored and free arrangements in this interval. (Each of these properties is known to imply an integral factorization of the characteristic polynomial.) It also provides the first example of an interesting family of hyperplane arrangements for which the notions of "factored" and "inductively factored" have a simple combinatorial characterization.

2. Background

2.1. **Arrangements of Hyperplanes.**

The standard reference on arrangements of hyperplanes is the book by Orlik and Terao [OT]. Let $\mathbf{k}$ be a field not of characteristic two. A *hyperplane* H in $\mathbf{k}^l$ is a codimension-one linear subspace of $\mathbf{k}^l$. A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of hyperplanes in $\mathbf{k}^l$. To emphasize the dimension of $\mathbf{k}^l$, we will refer to $\mathcal{A}$ as an *l-arrangement*.

For each arrangement $\mathcal{A}$, let $L(\mathcal{A})$ be the set of intersections of subsets of hyperplanes in $\mathcal{A}$. Define a partial order on $L(\mathcal{A})$ by

$$X \leq Y \quad \text{if and only if} \quad Y \subseteq X.$$

Then $L(\mathcal{A})$ is a *geometric lattice* which has the ambient vector space $\mathbf{k}^l$ as the unique minimal element. As a geometric lattice, $L(\mathcal{A})$ is ranked, and the *rank* $r(\mathcal{A})$ of an arrangement $\mathcal{A}$ is defined to be the rank of $L(\mathcal{A})$. Define the *characteristic polynomial* of $\mathcal{A}$ by

$$\chi(\mathcal{A}, t) = \sum_{X \in L} \mu(\mathbf{k}^l, X) t^{\dim(X)}.$$

Here μ is the *Möbius function* on $L(\mathcal{A})$.

A partial answer was provided by Stanley [St1] in 1971, in the form of *supersolvability*. This is a property of some arrangements, through their geometric lattices, that implies integral factorization. In 1981, Terao [Te1] showed that a larger class of arrangements, called *free* arrangements, also satisfies this property. Moreover, he provided an algebraic interpretation of the roots of $\chi(\mathcal{A}, t)$ for arrangements in this class. In 1992, Terao [Te2] defined *factored* hyperplane arrangements, which also have this property. A short time later, Jambu and Paris [JP] discovered a very well behaved subclass of both the free and the factored arrangements, called *inductively factored* arrangements, which properly includes all supersolvable arrangements.

To define these classes of arrangements precisely requires a bit more terminology. For $X \in L(\mathcal{A})$, define the *localization* of $\mathcal{A}$ to X to be

$$\mathcal{A}_X = \{H \in \mathcal{A} | X \subseteq H\}.$$

Note that if T is the unique maximal element in $L(\mathcal{A})$, then $\mathcal{A}$ itself is the localization of $\mathcal{A}$ to T. For $X \in L(\mathcal{A})$, define the *restriction* of $\mathcal{A}$ to X to be the following arrangement of hyperplanes within the vector subspace X:

$$\mathcal{A}^X = \{X \cap H | H \in \mathcal{A} - \mathcal{A}_X\}.$$

An l-arrangement $\mathcal{A}$ of rank r is *supersolvable* if it is possible to define an ordered partition $\pi = (\pi_1, \dots, \pi_r)$ of its hyperplanes such that the subarrangement $\mathcal{A}_{r-1} = \cup_{k=1}^{r-1} \pi_k$ is supersolvable of rank $r-1$ and the intersection of any two hyperplanes in π_r is contained in some hyperplane in $\mathcal{A}_{r-1}$. Any independent arrangement is supersolvable. If $l > r$, then trivially augment π with empty blocks $(\pi_{r+1}, \dots, \pi_l)$. Stanley's result can be stated for arrangements as:

Theorem 2.1. *[St1 Theorem 2] If $\mathcal{A}$ is supersolvable, then $\chi(\mathcal{A}, t) = \prod_{i=1}^{l}(t - b_i)$ where $b_i = |\pi_i|$.*

To define a factored arrangement, let $\pi = \{\pi_1, \dots, \pi_s\}$ partition the hyperplanes of $\mathcal{A}$. Note that here the blocks of π are unordered, in contrast to partitions of supersolvable arrangements. A collection of hyperplanes $\mathcal{H} = \{H_i\}_{i=1}^{s}$ in $\mathcal{A}$ is called a *section* of π if $H_i \in \pi_i$ for all $1 \leq i \leq s$. A partition π is called *independent* if every section of π is independent; that is, $\cap_{i=1}^{s} H_i$ is a subspace of codimension s for every section $\mathcal{H}$ of π. Note that, in particular, this condition implies $r(\mathcal{A}) \geq s$. For $X \in L(\mathcal{A})$, define the *induced partition* π_X of $\mathcal{A}_X$ to be the partition which has as its blocks the nonempty subsets $\pi_i \cap \mathcal{A}_X$.

A partition $\pi = \{\pi_1, \dots, \pi_s\}$ of $\mathcal{A}$ is called a *factorization* if

a) π is independent, and
b) if $X \in L(\mathcal{A}) - \{\mathbf{k}^l\}$, then the induced partition π_X contains a block which is a singleton. In particular, $|\pi_i| = 1$ for some i.

These conditions together imply $r(\mathcal{A}) \leq s$. Thus if $\mathcal{A}$ has a factorization $\pi = \{\pi_1, \dots, \pi_s\}$, then necessarily $s = r(\mathcal{A})$. If $\mathcal{A}$ has a factorization, then $\mathcal{A}$ is *factored*. Terao [Te2 Corollary 2.9] showed that, for a factored arrangement $\mathcal{A}$, $\chi(\mathcal{A}, t)$ factors exactly as in Theorem 2.1.

Supersolvable arrangements form a subclass of another important class of arrangements, the *free* arrangements. We will not define free arrangements precisely here, as the definition is not necessary for the results which follow. Informally, an arrangement $\mathcal{A}$ is *free* if a certain module associated with $\mathcal{A}$ is free over the polynomial ring $\mathbf{k}[x_1, \dots, x_l]$.

Every such free module has a homogeneous basis, and the degrees of the generators are called the *exponents* of $\mathcal{A}$. The multiset of exponents is denoted by $\exp\mathcal{A}$. Terao showed [OT Theorem 4.137] that if $\mathcal{A}$ is free, the exponents of $\mathcal{A}$ are the roots of $\chi(\mathcal{A}, t)$. For the remainder of this paper, the term "exponents" will refer to the roots of $\chi(\mathcal{A}, t)$ whenever $\mathcal{A}$ is free, factored, or supersolvable.

There is a special class of factored arrangements which are also free. These are the *inductively factored* arrangements, which are the main focus of this paper. Let $\mathcal{A}$ be factored with factorization π. Given a hyperplane $H_0 \in \pi_j$, define the *restriction of* π *to* H_0 to be $\pi'' = \{\pi''_1, \pi''_2, \dots, \pi''_{j-1}, \pi''_{j+1}, \dots, \pi''_l\}$, where $\pi''_i = \{H \cap H_0 | H \in \pi_i\}$ for $i \neq j$, $1 \leq i \leq l$. A factorization $\pi = \{\pi_1, \dots, \pi_l\}$ is said to be an *inductive factorization* (with respect to H_0) if there exists a hyperplane $H_0 \in \mathcal{A}$ such that the nonempty subsets $\pi_i \cap \mathcal{A}'$ form an inductive factorization of $\mathcal{A}' = \mathcal{A} - \{H_0\}$ and π'' is an inductive factorization of $\mathcal{A}'' = \mathcal{A}^{H_0}$. As with supersolvable arrangements, any independent arrangement is inductively factored.

Below is a diagram of implications illustrating the relationships between these special types of arrangements.

$$\begin{array}{ccccc} & & \text{supersolvable} & & \\ & & \Downarrow & & \\ \text{factored} & \Leftarrow & \text{inductively factored} & \Rightarrow & \text{free} \end{array}$$

The implication "$\mathcal{A}$ supersolvable implies $\mathcal{A}$ inductively factored" follows by induction and Theorem 2.2 below. The implication "$\mathcal{A}$ inductively factored implies $\mathcal{A}$ free" is given in [JP Proposition 2.2].

It follows immediately from the definition of a factored arrangement that if $\mathcal{A}$ is factored with factorization π, then for all $X \in L(\mathcal{A})$, the localization $\mathcal{A}_X$ is factored with factorization π_X. This fact will be quite useful in the sequel. A similar result for supersolvable arrangements follows from

Theorem 2.2. *[St2 Proposition 3.2] Let $\mathcal{A}$ be supersolvable. Then $\mathcal{A}_X$ and $\mathcal{A}^X$ are each supersolvable for all $X \in L(\mathcal{A})$.*

Orlik and Terao give a similar result for free arrangements:

Theorem 2.3. *[OT Theorem 4.37] If $\mathcal{A}$ is free, then $\mathcal{A}_X$ is free for all $X \in L(\mathcal{A})$.*

Edelman and Reiner [ER] found examples showing that it is not true, in general, that $\mathcal{A}$ free implies $\mathcal{A}^X$ free.

2.2. **Signed-Graphic Arrangements of Hyperplanes.**

This section contains the definitions and terminology for graphic and signed-graphic arrangements. For a simple graph $G = (V, E)$ and S a subset of V, let G_S denote the induced subgraph of G on the vertex set S. For any vertex $v \in V$, let N_v denote the neighborhood of v in G.

A signed graph $G = (G^+, G^-, L)$ consists of a simple graph $G^+ = (V, E^+)$, a simple graph $G^- = (V, E^-)$ on the same vertex set V, and a subset L (called the *loop set*) of the vertex set V. One may picture G^+ as a set of edges on V, each carrying the sign $+$, and G^- as a set of edges on V, each carrying the sign $-$. Since the loop set L is a subset of the vertex set, one may picture its elements as loops on the appropriate vertices. An example of a signed graph is given in Figure 1 below. It will be convenient to abuse notation and write $x_i = x_j$ for the hyperplane $\ker(x_i - x_j)$, and similarly for other forms. Given a signed graph $G = (G^+, G^-, L)$, define $\mathcal{A}(G)$ in $\mathbf{k}^n$ as follows:

$$\begin{array}{rcc} x_i = x_j \in \mathcal{A}(G) & \text{if} & \{v_i, v_j\} \in G^+. \\ x_i = -x_j \in \mathcal{A}(G) & \text{if} & \{v_i, v_j\} \in G^-. \\ x_i = 0 \in \mathcal{A}(G) & \text{if} & v_i \in L. \end{array}$$

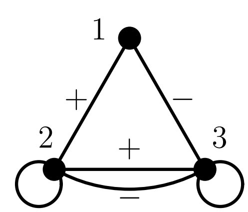

FIGURE 1

The signed graph in Figure 1 has $G^+ = \{\{1, 2\}, \{2, 3\}\}$, $G^- = \{\{1, 3\}, \{2, 3\}\}$, and $L = \{2, 3\}$. G thus corresponds to the arrangement of hyperplanes

$$\mathcal{A}(G) = \{x_1 = x_2,\ \ x_2 = x_3,\ \ x_1 = -x_3,\ \ x_2 = -x_3,\ \ x_2 = 0,\ \ x_3 = 0\}.$$

We will identify a signed-graphic hyperplane arrangement $\mathcal{A}(G)$ with its corresponding signed graph G, using the term "hyperplane" interchangeably with "edge" or "loop." For a given signed graph $G = G(\mathcal{A})$ and distinguished hyperplane H_0, we let G', G'' be the signed graphs corresponding to $\mathcal{A}'$, $\mathcal{A}''$, respectively. Also, we will say that a signed graph G is free, factored, inductively factored or supersolvable if $\mathcal{A}(G)$ is.

3. Arrangements in the interval $[A_{n-1}, B_n]$

Since all signed graphs whose arrangements are in the interval $[A_{n-1}, B_n]$ are of the form $G = (K_n^+, G^-, L)$, we will call them and the corresponding arrangements *positively complete*. Edelman and Reiner [ER] have characterized free and supersolvable positively complete arrangements. We will show that the subclasses of factored and inductively factored arrangements in this interval coincide, and provide a signed-graphic characterization of such arrangements. A bit more background is necessary before proceeding.

First, a general characterization of supersolvable signed graphs is necessary. Such a characterization has been provided by Zaslavsky [Za2]. A vertex v of a signed graph is a *simplicial vertex* if the intersection of every pair of edges (considered as hyperplanes) incident to v is contained in some edge (hyperplane) which is not incident to v. A signed graph G is a *simplicial extension* of an induced subgraph F of G if V can be partitioned into subsets S_1, S_2 in such a way that $F = G_{S_2}$ and a linear order $(v_1, \ldots, v_m)$ exists on the vertices in S_1 such that each v_i is a simplicial vertex in $G_{S_2 \cup \{v_1, \ldots, v_i\}}$. In the case where F is the empty graph (and thus $V = S_1$), the order $(v_1, \ldots, v_n)$ of the vertices of G is a *reverse vertex elimination order* on G (where the vertices are eliminated in **reverse** order — first v_n, then v_{n-1}, etc.). Zaslavsky's characterization of supersolvable signed graphs is as follows:

Theorem 3.1. *[Za2, Theorem 2] A signed graph G is supersolvable if and only if one of the following characterizations holds:*

- a) *G has a vertex elimination order.*
- b) *G is a simplicial extension of one of the following:*
  - i) *The D_3 graph (shown in Figure 2).*
  - ii) *A signed graph F in which all edges in F^- are incident to a single vertex v, the set of neighbors of v in F^- induces a complete subgraph in F^+, and F^+ has a vertex elimination order.*

It is clear that if case a) holds, then the partition $\pi = (\pi_1, \ldots, \pi_n)$ of the hyperplanes of $\mathcal{A}(G)$ which proves supersolvability is obtained by defining π_i to be the set of all edges incident to v_i in $G_{\{v_1, \ldots, v_i\}}$, including the loop on v_i if it exists. Since successive supersolvable subgraphs are obtained by ripping off simplicial vertices, the vertex elimination order (or equivalently, the partition π) will be called a *ripping order* on G. Note that the blocks π_i are removed in **reverse** order, in accordance with the definition of supersolvable arrangements.

In case b), for both the D_3 graph and the signed graph F described in the theorem, the first block removed in the ripping order consists of all negative edges.

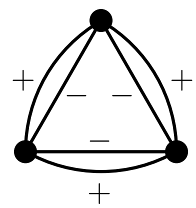

FIGURE 2. The D_3 arrangement

The final bit of background is a brief discussion of *threshold graphs.* Chvátal and Hammer [CH] provide the following useful theorem characterizing threshold graphs:

Theorem 3.2. *For any graph G, the following are equivalent:*

a) *G is a threshold graph.*
b) *G does not contain any of the graphs in Figure 3 as induced subgraphs.*
c) *For all vertices $v_i, v_j \in V$, if deg$(v_i) \geq$ deg(v_j), then $N_{v_i} \supseteq N_{v_j} - \{v_i\}$.*

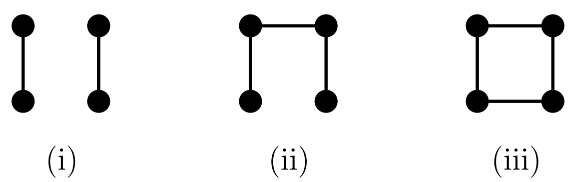

FIGURE 3

Since all positively complete signed graphs contain K_n^+, the pair (G^-, L) completely describes any positively complete signed graph G. Thus only G^- and the loop set L will be shown in the diagrams. Let L be the complement of L in V. We call G_L the *looped skeleton* of G and G_L the *loopless skeleton* of G. Also, the *degree* of a vertex v will mean its degree in the graph G^-. A linear ordering $(v_1, v_2, \ldots, v_n)$ such that

$$\deg(v_1) \geq \deg(v_2) \geq \cdots \geq \deg(v_n)$$

is a *degree order* on the vertices of G^-.

Theorem 3.3. *Let $G = (K_n^+, G^-, L)$ be a positively complete signed graph on n vertices. Then*

a) *G is free if and only if G^- is a threshold graph and L is an initial segment for some degree order on the vertices of G^-.*

b) *The following are equivalent:*

  i) *G is inductively factored.*

  ii) *G is factored.*

  iii) *G is free and one of the following holds:*

    E1) *G_L contains no edges, in which case G is supersolvable (see part c), below).*

    E2) *G_L contains exactly one edge $\{v_i, v_j\}$, and the vertices of G_L together with v_i and v_j are the vertices of a complete subgraph of G^-.*

    E3) *$L = \emptyset$ and $G^- = G_L$ is one of the graphs given in Figure 5.*

c) *G is supersolvable if and only if G is free* and *either*

  i) *G_L^- contains no edges, or*

  ii) *G^- is one of the graphs shown in Figure 4.*

(iv) (v)

FIGURE 4

Part a) of the theorem appears in [ER, Theorem 4.6], and part c) appears in [ER, Theorem 4.15]. Before proceeding to prove part b), note that any induced subgraph G_S of a signed graph G corresponds to a subarrangement $\mathcal{A}(G_S)$ of the arrangement $\mathcal{A}(G)$ which is the localization of $\mathcal{A}(G)$ to the subspace $\cap_{\{H \in \mathcal{A}(G_S)\}} H$. Thus Theorems 2.2 and 2.3 and the remarks immediately preceding them imply

Proposition 3.4. *If G is such that $\mathcal{A}(G)$ is a supersolvable (resp. free, factored) arrangement, then for any induced subgraph G_S of G, the arrangement $\mathcal{A}(G_S)$ is also supersolvable (resp. free, factored).*

Edelman and Reiner [ER] proved the necessity of the characterization in part a) of the theorem by showing that any signed graph which fails to meet both of the given conditions has a nonfactoring characteristic polynomial. Thus if a positively complete signed graph G is factored, it is also necessarily free.

We now proceed to prove part b) of the theorem. The implication i) $\Rightarrow$ ii) is trivial. To prove the implication ii) $\Rightarrow$ iii), suppose G is factored. G is free by the above observations, so G^- must be a threshold graph and $(v_1, \dots, v_n)$ a degree order on vertices such that L is an initial segment. If G_L contains no edges, then G is supersolvable by part c) of the theorem.

Now suppose G_L contains exactly one edge.

Lemma 3.5. *Let G be a free positively complete graph on n vertices such that G_L has exactly one edge on vertices $\{v_i, v_j\}$. Then:*

a) *Neither v_i nor v_j is adjacent to any other vertex in G_L.*
b) *Each of v_i, v_j is adjacent to every vertex in G_L.*
c) *$deg(v_i) = deg(v_j) > deg(v_k)$ for every $v_k \in L - \{v_i, v_j\}$.*
d) *$L \cup \{v_i, v_j\}$ induces a complete subgraph of G^-.*

Proof: Without loss of generality, $\deg(v_i) \geq \deg(v_j)$. Part a) follows immediately from the hypothesis that $\{v_i, v_j\}$ is the only edge in G_L. Suppose v_k is a vertex in G^- such that $\deg(v_k) \geq \deg(v_i) \geq \deg(v_j)$. Since G is free, part a) of the theorem shows that all vertices in L satisfy this inequality. By Theorem 3.2 part c), these inequalities imply

$$N_{v_k} \supseteq N_{v_i} - \{v_k\} \qquad \text{and} \qquad N_{v_k} \supseteq N_{v_j} - \{v_k\}.$$

Since $v_i \in N_{v_j} - \{v_k\}$ and $v_j \in N_{v_i} - \{v_k\}$, each of v_i, v_j is adjacent to v_k. This proves part b) of the lemma and also, in view of part a), that no loopless vertex v_k satisfies $\deg(v_k) \geq \deg(v_i)$. This observation, together with parts a) and b) of the lemma, proves part c). Furthermore, a similar argument shows that since $\deg(v_k) \geq \deg(v_i)$ for all $v_k \in L$, all vertices of G_L must be pairwise adjacent. In particular, the vertices v_i and v_j, together with the vertices of G_L, are the vertices of a complete subgraph of G^-.$\square$

Finally, suppose that G_L contains two or more edges.

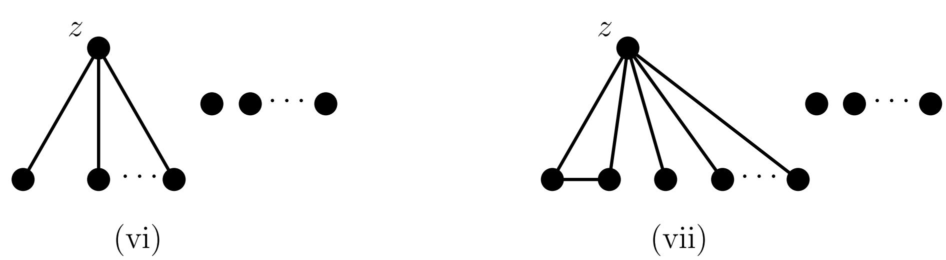

FIGURE 5

(1) We first demonstrate that if two edges in G_L are $\{v_1, v_2\}$ and $\{v_3, v_4\}$ with v_1, v_2, v_3, v_4 distinct, then G_L must look like graph (vii)

in Figure 5. The localization to the subspace $x_1 = x_2 = x_3 = x_4 = 0$ cannot look like one of the graphs in Figure 3, since G^- is a threshold graph. We will argue that neither of the other possibilities given in Figure 6 is factored.

(viii) (ix)

FIGURE 6

Graph (ix) is the D_4-arrangement, and it is well known that this arrangement is not factored. One may prove this by noting that $\chi(D_4, t)$ factors with exponents $\{1, 3, 3, 5\}$. It follows from [Te2 Corollary 2.9] that if D_4 has a factorization $\pi = \{\pi_1, \dots \pi_4\}$, then π must have block sizes $\{1, 3, 3, 5\}$. It is a routine matter to verify that any π with the given block sizes must violate one of the conditions for a factorization. Using the same sort of ad hoc argument, one can also show that graph (viii) is not factored (the exponents are $\{1, 3, 3, 4\}$). Therefore, by Proposition 3.4, the localization to $x_1 = x_2 = x_3 = x_4 = 0$ cannot look like any of graphs (i – iii) or (viii) or (ix). Since the same argument holds for any pair of nonadjacent edges, G_L can only be the signed graph shown in (vii).

(2) If each pair of edges of G_L share a common vertex, then again some routine checking shows that G_L must look like one of the graphs in Figure 5.

(3) Next, consider G_L. Suppose that G_L contains some vertex y. Let z be a vertex of greatest degree in G_L. Since G is free, it follows by part a) of the theorem that $\deg(z) \leq \deg(y)$. By part c) of Theorem 3.2, $N_y \supseteq N_z - \{y\}$.

By assumption, G_L contains at least two edges, and it is shown above that G_L is one of the graphs in Figure 5. If y and z are *not* adjacent in G^- but $N_y \supseteq N_z - \{y\}$, then since z has at least two loopless neighbors, the localization to $\{y\} \cup \{z\} \cup N_z$ is not free, hence not factored. Therefore y and z must be adjacent in G^-. But then G^- would have to contain an induced subgraph like one of those in Figure 7.

Again, ad hoc arguments of the kind used above show that these graphs are not factored, hence G^- cannot contain them as induced subgraphs (graph (x) has exponents $\{1, 3, 4, 4\}$ and (xi) has exponents

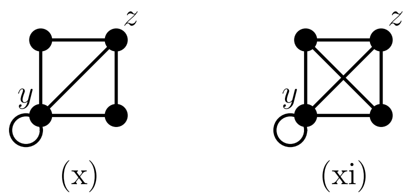

FIGURE 7

$\{1,3,4,5\}$). Therefore $L=\emptyset$, so G^- must look like one of the graphs shown in Figure 5.

The remainder of the proof is somewhat tedious. Readers who do not wish to be bored by details should skip ahead to page 14.

iii) $\Rightarrow$ i): If G_L contains no edges, then G is supersolvable and hence inductively factored.

If the condition in iii-E2) holds, then let $(v_1, v_2, \ldots, v_n)$ be a degree order on the vertices of G^- in which L is an initial segment. Lemma 3.5 shows that the unique loopless edge is $\{v_i, v_{i+1}\}$ for some i. Partition the edges of G (including positive edges and loops) as follows:

$$\begin{aligned} \pi_q &= \{\text{all edges from } v_q \text{ to } v_r, \text{ where } r \leq q\}, q \neq i, i+1, \\ \pi_{i+1} &= \{\text{all edges from } v_{i+1} \text{ to } v_r, \text{ where } r < i\} \cup \{x_i = x_{i+1}\}, \\ \pi_i &= \{\text{all edges from } v_i \text{ to } v_r, \text{ where } r < i\} \cup \{x_i = -x_{i+1}\}. \end{aligned}$$

Note that the description of π_q for $q \neq i, i+1$ includes the loops from v_k to v_k for $v_k \in G_L$. We will show that π is an inductive factorization of G with distinguished hyperplane $x_i = -x_{i+1}$. Let G' and G'' denote the signed graphs corresponding to $\mathcal{A}'$, and $\mathcal{A}''$, respectively.

First, it is easy to check that π' is a ripping order for G', so G' is supersolvable and consequently inductively factored by π'. In particular, Lemma 3.5 part d) implies that every vertex of G_L other than v_i and v_{i+1} is simplicial.

The signed graph G'' is obtained from G by identifying the vertices v_i and v_{i+1} (ignoring any multiple copies of edges from the new vertex $v_{i,i+1}$ to other vertices), adding the edge $\{v_{i,i+1}, v_q\}$ for every vertex $v_q \in G_L$ distinct from v_i and v_{i+1}, and adding a loop to $v_{i,i+1}$. One can check that the partition π'' is a ripping order for G'', and thus G'' is inductively factored with factorization π''.

It still remains to prove that π is a factorization of G. We first show that π is independent. Since π' is a ripping order, and is therefore independent, one need only consider sections of π which contain the edge $x_i = -x_{i+1}$. Suppose $\mathcal{H} = \{H_r\}$ is such a section, and suppose $\mathcal{H}$ is dependent. This implies that $x_i = -x_{i+1}$ contains $\cap_{H_r \in \mathcal{H}'} H_r$, where

$\mathcal{H}'$ is $\mathcal{H}$ with $x_i = -x_{i+1}$ removed. This can only happen if $\cap_{H_r \in \mathcal{H}'} H_r$ forces the relation $x_i = -x_{i+1}$. However, since $\mathcal{H}$ contains no element of π_i other than $x_i = -x_{i+1}$, the index i can only be represented in $\mathcal{H}'$ if $\mathcal{H}'$ contains the hyperplane $x_{q_1} = x_i \in \pi_{q_1}$ for some $q_1 \geq i+1$ (it cannot be $x_{q_1} = -x_i$ since $\{v_i, v_{i+1}\}$ is the only loopless edge in G^- and v_k is loopless for all $k > i$). Since $\mathcal{H}$ is a section and therefore does not contain any other hyperplane in π_{q_1}, the index q_1 occurs in $\mathcal{H}$ again only if $\mathcal{H}$ contains $x_{q_1} = x_{q_2}$ for some $q_2 > q_1$. This process may be iterated, but must terminate in some block π_{q_m}. There may be several distinct such strings of equalities, but none will involve a negative sign, and hence none will force the relation $x_i = -x_{i+1}$. Consequently $x_i = -x_{i+1}$ cannot contain the intersection of the hyperplanes in $\mathcal{H}'$, and $\mathcal{H}$ is independent.

As for the singleton condition, suppose $X \in L(\mathcal{A}(G)) - \{K^n\}$. If X is not contained in $x_i = -x_{i+1}$, then $\pi_X = \pi'_X$ has a singleton block. If $x_i = -x_{i+1}$ is the **only** edge in π_i to contain X, then π_X has the singleton block $\{x_i = -x_{i+1}\}$. So suppose X is contained in $x_i = -x_{i+1}$ and some other hyperplane in π_i. Let r be the least index such that X is contained in $x_i = \pm x_r$.

Then also X is contained in $x_{i+1} = \mp x_r$. If this is the only hyperplane of π_{i+1} containing X, then π_X contains a singleton block. Otherwise, let r_1 be the least index, $r_1 \neq r$, such that X is contained in $x_{i+1} = \pm x_{r_1}$. Since then X is contained in $x_i = \mp x_{r_1}$ in π_i, necessarily $r < r_1$.

If $r_1 = i$, then X is contained in the intersection of $x_i = -x_{i+1}$, $x_i = x_{i+1}$, and $x_i = \pm x_r$, and so X is contained in the loop $x_r = 0$ in the block π_r. Again, if $x_r = 0$ is the only such hyperplane in π_r, then π_X has a singleton block. Otherwise, X is contained in $x_r = \pm x_{r_2}$ with $r_2 < r$. But then X is also contained in $x_i = \pm x_{r_2}$ in π_i, contradicting the choice of r.

If $r_1 \neq i$, then X is contained in the intersection of $x_{i+1} = \mp x_r$ and $x_{i+1} = \pm x_{r_1}$, and so X is contained in $x_r = \pm x_{r_1}$ in π_{r_1}. If $x_r = \pm x_{r_1}$ is the only hyperplane in π_{r_1} that contains X, then π_X contains a singleton block. Otherwise, there exists some $r_2 < r_1$ such that $x_{r_1} = \pm x_{r_2}$ also contains X. However, since this in turn implies that $x_{i+1} = \pm x_{r_2}$ contains X, we must conclude that$r_2 = r$ by our choice or r_1. In this case, X is contained in both hyperplanes $x_{r_1} = x_r$ and $x_{r_1} = -x_r$ and therefore in the hyperplane $x_r = 0$ in π_r. Using the argument from the preceding paragraph, we conlcude that π_X contains a singleton block for any choice of X.

Thus π satisfies the singleton condition and consequently is a factorization. Since the deletion G' and restriction G'' on $x_i = -x_{i+1}$ are each inductively factored by π' and π'' respectively, G is inductively factored.

It only remains to show that the signed graphs (vi) and (vii) are inductively factored. It is clear by part c) of the theorem that graph (vi) is inductively factored, since it is supersolvable. For case (vii), we show that the graph in Figure 8

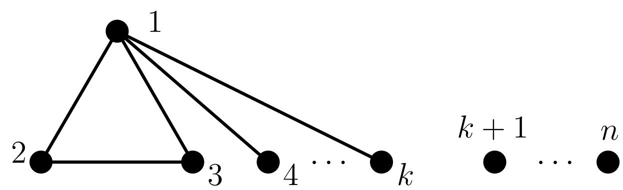

FIGURE 8

is factored with factorization as follows:

$$\begin{aligned}
\pi_1 &= \{x_2 = x_3\} \\
\pi_2 &= \{x_1 = x_2,\, x_1 = x_3,\, x_2 = -x_3\} \\
\pi_3 &= \{x_1 = -x_i \,:\, 2 \le i \le k\} \\
\pi_p &= \{x_p = x_r \,:\, r < p\} \quad \text{for all} \quad 4 \le p \le n.
\end{aligned}$$

Note that a supersolvable signed graph is obtained by removing the hyperplane $x_2 = -x_3$, and in fact the factorization given is a ripping order for the resulting signed graph if we reorder the blocks so that the block π_3 is the first removed, and then continue with blocks $\pi_n, \pi_{n-1}, \ldots, \pi_2 - \{x_2 = -x_3\}, \pi_1$ as usual. Let π' denote this ripping order.

Thus to show that π is independent, it is sufficient to consider only those sections $\mathcal{H}$ which contain the hyperplane $x_2 = -x_3$ from π_2. As before, in order for $\mathcal{H}$ to be dependent, the hyperplanes in $\mathcal{H}' = \mathcal{H} - \{x_2 = -x_3\}$ must force the relation $x_2 = -x_3$.

Suppose $x_1 = -x_r$ is the unique hyperplane in $\mathcal{H}'$ from the block π_3. Since this is the only hyperplane in $\mathcal{H}'$ which involves a negative sign, then without loss of generality the relation $x_2 = -x_3$ can be obtained only if there exist hyperplanes in $\mathcal{H}'$ which give the chains of equalities

$$\begin{aligned}
x_2 = x_{q_1} = x_{q_2} = \cdots = x_1 \\
x_r = x_{r_1} = x_{r_2} = \cdots = x_3
\end{aligned}$$

(the multisets of indices $\{\mathrm{q}_1, q_2, \ldots\} and \{r_1, r_2, \ldots\} need not be disjoint)$.

However, the first of these chains of equalities cannot occur, for if it did, the section obtained by replacing $x_2 = -x_3$ with $x_1 = x_2$ from π_2

would also fail to be independent, contradicting the fact that π' is a ripping order and thus independent. Therefore π is also independent.

To verify the singleton condition, it is again sufficient to consider only those subspaces X contained in $x_2 = -x_3$. If no other hyperplane of π_2 contains X, then π_X contains a singleton block. If **all** hyperplanes in π_2 contain X, then X is also contained in $x_2 = x_3$, and thus π_X has a singleton block. So without loss of generality, assume X is contained only in $x_2 = -x_3$ and $x_1 = x_2$ in π_2.

Then X is contained in $x_1 = -x_3$ in π_3. If some other hyperplane of π_3 contains X, let r be the least index, $r > 3$, such that X is contained in $x_1 = -x_r$. If X is contained in $x_1 = -x_2$, then X is also contained in $x_1 = x_3$, which is a contradiction. Then $x_r = x_3$ must be the only hyperplane of π_r to contain X, so π_X has a singleton block.

One can further check that the following is a sequence of hyperplanes which makes the given factorization an *inductive* factorization.

a) Remove all hyperplanes of the form $x_i = x_n$ for $i < n$, then all hyperplanes of the form $x_i = x_{n-1}$ for $i < n-1$, and so forth until all hyperplanes involving isolated vertices have been removed.
b) Remove the hyperplane $x_1 = -x_k$ and all hyperplanes of the form $x_i = x_k$ with $i < k$, and continue in this manner until only the D_3 arrangement remains.

□

4. Conclusions and conjectures

The result in this paper, as well as the results in [ER] and [Za2], is a partial attempt to generalize Stanley's very efficient characterization of free, factored, inductively factored and supersolvable subarrangements of A_{n-1}. In terms of signed graphs, these arrangements correspond to subarrangements of $(K_n^+, \emptyset, \emptyset)$, and are called *graphic* arrangements. Stanley's result is

Theorem 4.1. *For graphic arrangements of hyperplanes, the classes of free, factored, inductively factored and supersolvable arrangements coincide. An arrangement $\mathcal{A}(G)$ is in this class if and only if G is a* chordal *graph (that is, every circuit of length greater than three has a chord).*

The proof that all chordal graphs are supersolvable (hence inductively factored, factored, free) appears in [St 2]. It is difficult to find a proof in the literature that "free implies chordal," but it is a fairly routine matter to verify that any chordless cycle of length four or greater has a nonfactoring characteristic polynomial.

With the assistance of P. H. Edelman and V. Reiner, we have used methods similar to those employed in this paper to prove *unsigned*-graphic characterizations for the free, factored, inductively factored and supersolvable arrangements in the following subclasses of $B_n = (K_n^+, K_n^-, \{1, 2, \ldots, n\})$:

a) Subarrangements of $(K_n^+, \emptyset, \{1, 2, \ldots, n\})$ (which are isomorphic to subarrangements of K_{n+1}).
b) Subarrangements of $(\emptyset, K_n^-, \{1, 2, \ldots, n\})$.
c) Subarrangements of B_n which contain $(\emptyset, K_n^-, \emptyset)$.
d) Arrangements (G^+, G^-, L) with $G^+ = G^-$.

In all of the above classes, the characterizations are much easier to prove and not as interesting as the results of Section 3. We therefore state without proof the following theorems.

Theorem 4.2. *For signed graphs which are subgraphs of* $(\emptyset, K_n^-, \{1, 2, \ldots, n\})$, *the classes of free, factored, inductively factored and supersolvable arrangements coincide. An arrangement* $\mathcal{A}(G)$ *is in this class if and only if* G *is one of the following types of signed graph:*

a) *G has exactly one cycle C of odd length, and G has no loops.*
b) *G is a tree in which G_L forms a subtree.*

Theorem 4.3. *Let* $G = (G^+, K_n^-, L)$ *be a signed graph on* n *vertices containing* K_n^-.

a) *G is free if and only if one of the following holds:*
   i) *G^+ is a complete graph.*
   ii) *There exists a vertex v such that G^+_{V-v} is complete and either $v \notin L$ or $L = \{1, 2, \ldots, n\}$.*
   iii) *G^+ consists of K_n with loops at all but three vertices and with the three loopless edges removed.*
b) *The following are equivalent:*
   i) *G is factored.*
   ii) *G is inductively factored.*
   iii) *Either G is free of types i) or ii) above, and G^+ has at most one loopless edge, or G is free of type iii) above.*
c) *G is supersolvable if and only if G is free of types i) or ii) above and G^+ has no loopless edges.*

Theorem 4.4. *Suppose* $G = (G^+, G^-, L)$ *is such that* $G^+ = G^-$ *and* $G \neq D_3$.

a) *G is free if and only if $G^+ = G^-$ is chordal and all loopless vertices of G are simplicial vertices of the subgraph G^+.*

b) *The following are equivalent:*
   i) *G is factored.*
   ii) *G is inductively factored.*
   iii) *either $G^+ = K_3$ or G is free and no two edges of G_L^+ are incident on a common vertex, or both.*

c) *G is supersolvable if and only if either $G^+ = K_3$ or G is free and G_L^+ has no edges, or both.*

Note that in every one of the above cases, the classes of factored and inductively factored arrangements coincide. This leads to the following

Conjecture 4.5. If G is any signed graph, then G is factored if and only if it is inductively factored.

Jambu and Paris [JP] state that they know of no arrangement of hyperplanes over $\mathbb{R}$ which is factored but not inductively factored, so the conjecture might very well be true for the class of all real hyperplane arrangements. Nevertheless, a proof might be more accessible within the restricted class of signed-graphic arrangements.

5. Acknowledgements

The author would like to acknowledge the helpful contributions of P. H. Edelman, M. Falk, and V. Reiner, and would like to thank the anonymous referee for many helpful suggestions.

References

[Ar] Arnold, V.I.: The cohomology ring of the colored braid group. Matematicheskie Zametki, **5**, 227-231 (1969): Mathematical Notes, **5**, 138-40 (1969)

[CH] Chvátal, V., Hammer, P.L.: Aggregation of inequalities in integer programming. Annals of Discrete Mathematics, **1**, 145-162 (1977)

[ER] Edelman, P.H., Reiner, V.: Free hyperplane arrangements between A_{n-1} and B_n. Mathematische Zeitschrift, **215**, 347-365 (1994)

[FaN] Fadell, E., Neuwirth, L.: Configuration spaces. Mathematica Scandinavica, **10**, 111-118 (1962)

[FoN] Fox, R.H., Neuwirth, L.: The braid groups. Mathematica Scandinavica, **10**, 119-126 (1962)

[FR] Falk, M., Randell, R.: The lower central series of a fiber-type arrangement. Inventiones Mathematicae, **82**, 77-88 (1985)

[Ha] Harary, F.: On the notion of balance of a signed graph. Michigan Mathematical Journal, **2**, 143-146 (1953-54)

[OT] Orlik, P., Terao, H.: Arrangements of Hyperplanes. (Grundlehren Math. Wiss. Bd. 300) Springer, New York, 1992

[JP] Jambu, M., Paris, L.: Combinatorics of inductively factored arrangements. European Journal of Combinatorics, **16**, 267-292 (1995)

[Sa] Saito, K.: On the uniformization of complements of discriminant loci. Conference Notes, Summer Institute, American Mathematical Society, Williamstown, 1975

[St1] Stanley, R.: Modular elements of geometric lattices. Algebra Universalis, **1**, 214-217 (1971)

[St2] _____: Supersolvable lattices. Algebra Universalis, **2**, 197-217 (1972)

[Te1] Terao, H.: Generalized exponents of a free arrangement of hyperplanes and Shephard-Todd-Brieskorn formula. Inventiones Mathematicae, **63**, 159-179 (1981)

[Te2] _____: Factorizations of Orlik-Solomon algebras. Advances in Mathematics, **91**, 45-53 (1992)

[Za1] Zaslavsky, T.: The geometry of root systems and signed graphs. American Mathematical Monthly **88**, 88-105 (1981)

[Za2] _____: Supersolvable frame-matroid and graphic-lift lattices, preprint.

Campus Box 7281, Hickory, North Carolina, 28603
E-mail address: `baileyd@mail.lrc.edu`