Math 4707 Intro to combinatorics and graph theory
Fall 2020, Vic Reiner
Final exam - Due Wednesday December 16, at our Canvas site

Instructions: There are 4 problems, worth 25 points each, totaling 100 points. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (25 points total) DNA strings can be thought of as ordered sequences of 4 possible nucleotides:

- A (=adenine)
- C (=cytosine)
- G (=guanine)
- T (=thymine).

For example, $CCGATAAGGGCTCA$ is such a sequence.

(a) (5 points) How many such sequences are there of length 800?

(b) (5 points) How many such sequences are there of length 800 in which there are no two equal letters in adjacent positions, that is, no adjacent AA, CC, GG or TT's?

(c) (5 points) How many such sequence are there of length 800 as in part (a), that have equally many A's as C's, and equally many G's as T's, but three times as many G's as A's.

(d) (10 points) How many such sequence are there of length 800 as in part (a), that use all four letters A, C, G, T at least once.

2. (25 points total) You are given a three dimensional polyhedron with all quadrangular (four-sided) faces and a total of e edges.

(a) (10 points) How many faces does it have, as a function of e?

(b) (5 points) Prove that e is even.

(c) (10 points) How many vertices does it have, as a function of e?

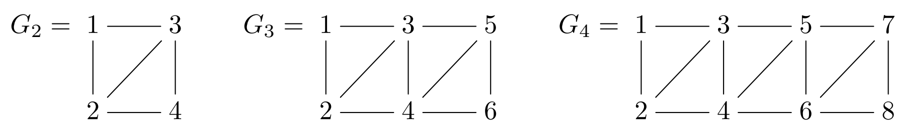
3. (25 points) Recall that we defined the Fibonacci numbers $F_0, F_1, F_2, \ldots$ by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = 0, F_1 = 1$.

Prove that F_n mod 4, the remainder of F_n upon division by 4, will repeat with period 6, meaning $F_n \equiv F_{n+6}$ mod 4 for all $n \geq 0$, and fill in the ?'s in this table with the correct remainders $0, 1, 2, 3$ mod 4:

n mod 6	F_n mod 4
0	?
1	?
2	?
3	?
4	?
5	?

You must prove your answers, of course.

4. (25 points total) Consider a family of graphs $G_2, G_3, G_4, \ldots$, where G_n has vertices $V = \{1, 2, \ldots, 2n-1, 2n\}$, and edges as shown below



Compute explicitly the *chromatic polynomial* $\chi(G_n, k)$ for ...

(a) (5 points) ... the case $n = 2$, that is G_2.

(b) (5 points) ... the case $n = 3$, that is G_3.

(c) (10 points) ... the general case of G_n for $n \geq 2$.

(d) (5 points) How many acyclic orientations are there for G_n with $n \geq 2$, as a function of n?