Math 4707 Intro to combinatorics and graph theory
Fall 2020, Vic Reiner
Midterm exam 1- Due Wednesday Oct 21, at our Canvas site

Instructions: There are 4 problems, worth 25 points each, totaling 100 points. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate, nor consult any sources that involve asking a human to solve the problem. The instructor is the only human source you are allowed to consult.

1. (25 points total)

(a) (5 points) How many rearrangements are there of the letters in the word "BOOKKEEPING"

(b) (5 points) When writing the answer to part (a) in *binary*, how many binary digits (bits) will there be exactly?

(c) (15 points) What is the probability that such a rearrangement as in part (a) has no identical letters **consecutive** (no "OO", "KK" nor "EE")?

2. (25 points total) You are supposed to choose 200 appetizers for a party, from 10 distinct varieties offered by the caterer.

(a) (10 points) How many ways are there for you to choose them? It only matters how many of each kind you order.

(b) (15 points) Suppose that exactly one of the 10 varieties is vegetarian, and you have been told that you must choose *at least* 50 out of the 200 appetizers from this variety. Now how many ways are there for you to choose the appetizers?

3. (25 points total) Recall that the Fibonacci numbers are defined by a recurrence $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, with initial conditions $F_0 = 0, F_1 = 1$. Recall also "$a \equiv b \bmod m$", read "a is congruent to b modulo m", means that a, b have the same remainder upon division by m, or equivalently that $a - b$ is a multiple of m.

(a) (15 points) After tabulating $F_0, F_1, F_2, \ldots, F_{24}$ and computing their remainders upon division by 3, fill in the blanks that make the following conjecture correct:

Conjecture: *The Fibonacci numbers have*

$$F_n \equiv \begin{cases} 0 \bmod 3 & \text{if } n \equiv \underline{?} \text{ or } \underline{?} \bmod 8 \\ 1 \bmod 3 & \text{if } n \equiv \underline{?} \text{ or } \underline{?} \text{ or } \underline{?} \bmod 8 \\ 2 \bmod 3 & \text{if } n \equiv \underline{?} \text{ or } \underline{?} \text{ or } \underline{?} \bmod 8 \end{cases}$$

(b) (10 points) Prove your conjecture.

4. (25 points total) Recall that we showed in lecture how to count the number of walks from $(0, 0)$ to (m, n) taking unit steps north or east that avoid passing through some point (a, b) where $0 \leq a \leq m$ and $0 \leq b \leq n$. Our answer was

$$\binom{m+n}{m} - \binom{a+b}{a}\binom{(m-a)+(n-b)}{m-a}$$

(a) (5 points) Show that this answer can be written as the determinant $\det(A)$ of the following 2×2 matrix A:

$$A = \begin{bmatrix} \binom{m+n}{m} & \binom{a+b}{a} \\ \binom{(m-a)+(n-b)}{m-a} & 1 \end{bmatrix}.$$

(b) (20 points) Prove that the number of such walks from $(0, 0)$ to (m, n) that avoid both points $(a_1, b_1), (a_2, b_2)$ where $0 \leq a_1 \leq a_2 \leq m$ and $0 \leq b_1 \leq b_2 \leq n$ can be written as the negative of the determinant $-\det(B)$ for the following 3×3 matrix B:

$$B = \begin{bmatrix} \binom{m+n}{m} & \binom{a_2+b_2}{a_2} & \binom{a_1+b_1}{a_1} \\ \binom{(m-a_1)+(n-b_1)}{m-a_1} & \binom{(a_2-a_1)+(b_2-b_1)}{a_2-a_1} & 1 \\ \binom{(m-a_2)+(n-b_2)}{m-a_2} & 1 & 0 \end{bmatrix}.$$