Math 4707 Intro to combinatorics and graph theory
Fall 2020, Vic Reiner
Midterm exam 2- Due Wednesday November 25, at our Canvas site

Instructions: There are 5 problems, worth 20 points each, totaling 100 points. This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total) Recall that a *forest* is a graph containing no cycles, that a *tree* is a connected forest, and a *leaf* is a vertex of degree one.

(a) (10 points) Prove that a tree with at least one degree d vertex has at least d distinct leaves.

(b) (5 points) Prove that a tree T with n vertices has

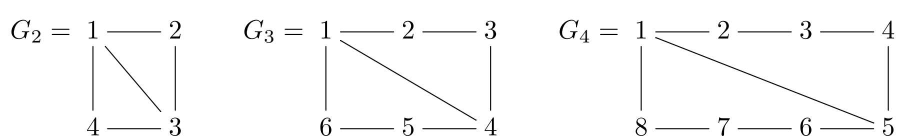
$$\sum_{v \in V} (\deg_T(v) - 1) = n - 2.$$

(c) (5 points) Given a forest with n vertices and c connected components, how many edges will it contain (as a function of n and c)? Prove your answer, of course.

2. (20 points total) For each $n = 2, 3, 4, \ldots$, define a graph $G_n = (V_n, E_n)$ having the following $2n$ vertices and$2n + 1$ edges

$$V_n := \{1, 2, 3, \ldots, 2n-1, 2n\},$$
$$E_n := \{\{1,2\}, \{2,3\}, \{3,4\}, \ldots, \{2n-1, 2n\}\} \cup \{\{1, 2n\}, \{1, n+1\}\}.$$

For example, G_2, G_3, G_4 are depicted here:



Calculate the number of spanning trees $\tau(G_n)$ for G_n as a function of n.

3. (20 points total; 10 points each part)
Let $G = (V, E)$ be bipartite graph, with vertex partition $V = X \sqcup Y$. Assume further that

- every x in X has the same degree $d_X \geq 1$, and
- every y in Y has the same degree $d_Y \geq 1$.

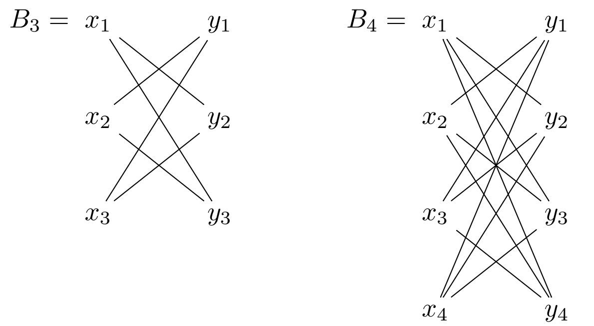
(a) Prove that $\frac{d_X}{d_Y} = \frac{|Y|}{|X|}$.

(b) Assuming without loss of generality that $d_X \geq d_Y$, show that there exists at least one matching $M \subset E$ with number of edges $|M| = |X|$.

4. (20 points) For each $n = 2, 3, 4, 5, \ldots$, define a bipartite graph B_n on vertex $V = X \sqcup Y$ where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ with edge set

$$E := \{\{x_i, y_j\} : i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, n, \text{ and } i \neq j\}$$

Therefore B_n has $2n$ vertices, $n(n-1)$ edges. For example, here are B_3, B_4:



Explain, with proof, exactly for which values of n in $\{2, 3, \ldots\}$ does the graph B_n contain ...

(a) ... a spanning tree?

(b) ... a closed Euler tour/circuit?

(c) ... a perfect matching?

(d) ... a closed Hamiltonian tour/circuit?

5. (20 points total) In dominoes, a standard package has $\binom{8}{2} = 28$ dominoes, each with two ends labelled by numbers from $\{0, 1, 2, 3, 4, 5, 6\}$, with each possible unordered pair of labels occurring once, so the $i - j$ domino is the same as the $j - i$ domino:

$0-0$,	$0-1$,	$0-2$,	$0-3$,	$0-4$,	$0-5$,	$0-6$,
	$1-1$,	$1-2$,	$1-3$,	$1-4$,	$1-5$,	$1-6$,
		$2-2$,	$2-3$,	$2-4$,	$2-5$,	$2-6$,
			$3-3$,	$3-4$,	$3-5$,	$3-6$,
				$4-4$,	$4-5$,	$4-6$,
					$5-5$,	$5-6$,
						$6-6$

The goal is to lay them out touching two-at-a-time end-to-end in one long cycle, but only touching at ends with matching labels, e.g. the $2-5$ and $5-4$ dominoes can touch at their ends labelled 5.

Prove this is possible, without exhibiting such a cycle explicitly, by proving this: given a similar pack of $\binom{n+2}{2}$ dominoes having ends labelled with unordered pairs from $\{0, 1, 2, \ldots, n\}$ then this goal is possible if and only if n is even.
(**Hint:** how does this relate to Euler tours in some graph?)