(145)

<u>PROPOSITION</u>: If $V \xrightarrow{T} V$ had matrix $\underset{\overset{\shortparallel}{(a_{ij}) \in F^{n\times n}}}{A}$ w.r.t. basis $\mathbb{B} = (v_1, \ldots, v_n)$.

(4.3.5)

then w.r.t. a <u>new</u> <u>basis</u> $\mathbb{B}' = (v_1', \ldots, v_n')$

that has $v_j' = \sum_{i=1}^{n} p_{ij} v_i$ for $P = (p_{ij}) \in GL_n(F)$ as change-of-basis matrix,

T will have matrix $A' = P^{-1}AP$ i.e. <u>conjugation</u> of A by P.
(similarity <u>transformation</u>)

12/10/2018 >

<u>proof</u>: Follows from what we did, but let's just do a sanity check.

We have $T(v_j) = \sum_{i=1}^{n} a_{ij} v_i$
for $j = 1, \ldots, n$

and also $v_j = \sum_{i=1}^{n} p^{-1}_{ij} v_i'$,

so
$$T(v_j') = T\left(\sum_{i=1}^{n} p_{ij} v_i\right) = \sum_{i=1}^{n} p_{ij} T(v_i)$$
$$= \sum_{i=1}^{n} p_{ij} \sum_{k=1}^{n} a_{ki} v_k$$
$$= \sum_{i=1}^{n} p_{ij} \sum_{k=1}^{n} a_{ki} \sum_{l=1}^{n} p^{-1}_{lk} v_l'$$
$$= \sum_{l=1}^{n} \underbrace{\left(\sum_{k=1}^{n} \sum_{i=1}^{n} p^{-1}_{lk} a_{ki} p_{ij}\right)}_{(P^{-1}AP)_{lj}} v_l', \text{ as desired } \blacksquare$$

So the idea of §4.4, 4.5, 4.6, 4.7 is to figure out how to make particularly useful changes-of-basis $\mathbb{B} \rightsquigarrow \mathbb{B}'$
so that $A \rightsquigarrow PAP^{-1}$ becomes particularly <u>simple</u>.

§4.4, 4.5 Eigenvectors & diagonalization

Q: When we can pick a basis $\mathbb{B}=(v_1,\ldots,v_n)$ for V that makes $V \xrightarrow{T} V$ having matrix that is diagonal

$$A = \begin{array}{c|c} & \begin{matrix} v_1 & v_2 & \cdots & v_n \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix} & \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \end{array} \; ?$$

A: This would mean $T(v_i)=\lambda_i v_i$ for $i=1,2,\ldots,n$,

that is, V has a basis $\mathbb{B}=(v_1,\ldots,v_n)$ of eigenvectors for T ..

DEF'N: One says $v \in V$ is an eigenvector for $T: V \to V$ with eigenvalue λ if $Tv=\lambda v$ (and $v \neq \underline{0}$!)

Thus we noted above T or its matrix A is diagonalizable,

meaning $\exists$ a basis $\mathbb{B}=(v_1,\ldots,v_n)$ for V which is diagonal $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

(or equivalently $\exists P \in GL_n(F)$ with $P^{-1}AP = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$)

$\Longleftrightarrow$ V has a basis of eigenvectors for T

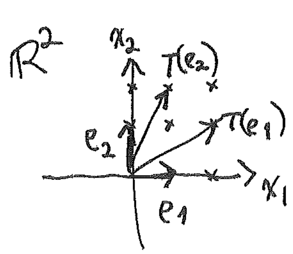
EXAMPLES: ① $\mathbb{R}^2 \xrightarrow{A=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \mathbb{R}^2$ has $v_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

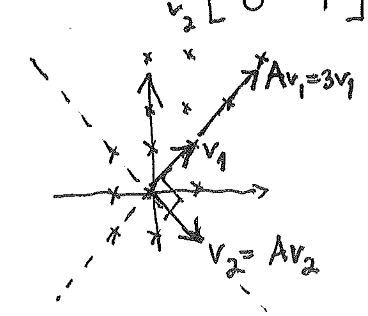
satisfying $Av_1=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 3 \\ 3 \end{bmatrix}=3\begin{bmatrix} 1 \\ 1 \end{bmatrix}=3v_1$

$Av_2=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}=1\cdot\begin{bmatrix} 1 \\ -1 \end{bmatrix}=1\cdot v_2$

and hence in the new basis $\mathbb{B}'=(v_1,v_2)$ one has $P=\begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \in GL_2(\mathbb{R})$

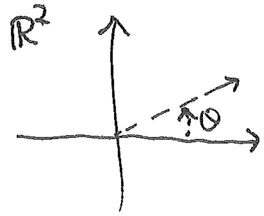
satisfying $A' = P^{-1}AP = \begin{array}{c|c} & \begin{matrix} v_1 & v_2 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \end{matrix} & \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \end{array}$. This makes A's action on $\mathbb{R}^2$ easier to understand:





② Sometimes $A \in F^{n\times n}$ is not diagonalizable until you pass to a bigger field $\bar{F} \supset F$ where eigenvalues/eigenvectors exist!

e.g. $\mathbb{R}^2 \xrightarrow[\underset{\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}}{\parallel}]{A_\theta \in \mathbb{R}^{2\times 2}} \mathbb{R}^2$ has no eigenvectors in $\mathbb{R}^2$ (unless $\theta = 0$ or π)



but if we think of $A_\theta \in \mathbb{C}^{2\times 2}$, then it becomes diagonalizable:

$$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

have
$$Av_1 = \begin{bmatrix} i\cos\theta - \sin\theta \\ i\sin\theta + \cos\theta \end{bmatrix} = \begin{bmatrix} ie^{i\theta} \\ e^{i\theta} \end{bmatrix} = e^{i\theta}\begin{bmatrix} i \\ 1 \end{bmatrix} = e^{i\theta} v_1$$

$$Av_2 = \begin{bmatrix} -i\cos\theta - \sin\theta \\ -i\sin\theta + \cos\theta \end{bmatrix} = \begin{bmatrix} -ie^{-i\theta} \\ e^{-i\theta} \end{bmatrix} = e^{-i\theta}\begin{bmatrix} -i \\ 1 \end{bmatrix} = e^{-i\theta} v_2$$

so $P = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \in GL_2(\mathbb{C})$ satisfies

$$P^{-1}AP = \begin{array}{c c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \end{array} & \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \end{array}, \text{ diagonal in } \mathbb{C}^{2\times 2}$$

③ Sometimes A will not be diagonalizable, even after enlarging F to $\bar{F}$

e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2} \subset \mathbb{C}^{2\times 2}$ is diagonalizable nowhere,

but at least its upper triangular

(so triangulizable, i.e. $P^{-1}AP = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Let's understand this better, with three goal theorems in mind:

THEOREM: Every linear map $\mathbb{C}^n \xrightarrow{T} \mathbb{C}^n$ can be triangularized: $T = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & \cdots & v_n \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \end{array} & \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \end{array}$
(PROP. 4.6.1)

THEOREM: "Most" linear maps $\mathbb{C}^n \xrightarrow{T} \mathbb{C}^n$ can be diagonalized,
(4.6.6) at least those for which $p_T(t) = \det(tI - \underset{\text{any matrix for } T}{A}) = (t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$ has distinct roots $\lambda_1, \ldots, \lambda_n$

("Spectral") THEOREM: If $A \in \mathbb{R}^{n\times n}$ is symmetric then $\exists$ real eigenvalues $\lambda_1, \ldots, \lambda_n$ and an orthogonal
(8.6.10) matrix $P = (P^T)^{-1}$ in $GL_n(\mathbb{R})$ with $\underset{P^TAP}{P^{-1}AP} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

(148) The key tool is that eigenvalues are roots of $p_A(t)$:

DEF'N: Given

$$\begin{tikzcd} V \arrow[r, "T"] & V \\ F^n \arrow[u] \arrow[r, "A"] & F^n \arrow[u] \end{tikzcd}$$

, its characteristic polynomial is

$$p_A(t) := \det(tI_n - A)$$

$$= \det \begin{bmatrix} t-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & t-a_{22} & & \\ \vdots & & \ddots & \\ -a_{n1} & & & t-a_{nn} \end{bmatrix}$$

$$= t^n - \underbrace{(a_{11}+\dots+a_{nn})}_{\text{Trace}(A)} t^{n-1} + \dots \cancel{\pm}(-1)^n \det A$$

a polynomial in t

12/12/2018

NOTE: $p_A(t)$ depends only on T, since if we change bases, we get $A' = P^{-1}AP$ with $\det(tI_n - P^{-1}AP)$

$$= \det(P^{-1}(tI_n)P - P^{-1}AP)$$

$$= \det(P^{-1} \cdot (tI_n - A) \cdot P)$$

$$= \det(P^{-1}) \cdot \det(tI_n - A) \det(P)$$

$$= \det(tI_n - A)$$

PROPOSITION: $\lambda \in F$ is an eigenvalue for T (say with an eigenvector $v \neq 0$) $Tv = \lambda v$ $\iff$ $t = \lambda$ is a root of $p_A(t) = 0$ (and $v \in \ker(\lambda I_n - A)$)

proof: $Tv = \lambda v$ for some $v \neq 0$ in V

$\iff$ $Av = \lambda v$ — " —

$\iff$ $\lambda v - Av = 0$ — " —

$\iff$ $(\lambda I_n - A)v = 0$ — " —

$\iff$ $\lambda I_n - A$ is noninvertible in $F^{n\times n}$

$\iff$ $\det(\lambda I_n - A) = 0$ i.e. $t = \lambda$ is a root of $\det(tI - A) =: p_A(t)$ ∎