<u>proof:</u> For (i), if P is invertible and $P^{-1} = (p^{-1}_{ij})$ then $\{v_1', \dots, v_n'\}$ span V

since

$$\sum_{i=1}^{n} p^{-1}_{ij} v_i' = \sum_{i=1}^{n} p^{-1}_{ij} \left(\sum_{k=1}^{n} p_{ki} v_k \right) = \sum_{k=1}^{n} \underbrace{\left(\sum_{i=1}^{n} p_{ki} p^{-1}_{ij} \right)}_{(P \cdot P^{-1})_{kj} = I_{kj}} v_k = v_j$$

and this is <u>reversible</u>: if $\{v_1', \dots, v_n'\}$ span V

then $\exists (q_{ij})$ with

$$v_j = \sum_{i=1}^{n} q_{ij} v_i' = \sum_{i=1}^{n} q_{ij} \left(\sum_{k=1}^{n} p_{ki} v_k \right) = \sum_{k=1}^{n} \underbrace{\left(\sum_{i=1}^{n} p_{ki} q_{ij} \right)}_{\text{must be } \left\{ \begin{smallmatrix} 1 & \text{if } k=j \\ 0 & \text{else} \end{smallmatrix} \right\}} v_k$$

since $v_1, \dots, v_n$ are independent

Thus $Q = (q_{ij}) = P^{-1}$.

So $\{v_1', \dots, v_n'\}$ span $V \iff P$ invertible

hence they're a basis $\iff P$ invertible (since $\dim V = n$)

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For (ii), if $v \in V$ has coordinates $\underline{x}$ for B

then $v = x_1 v_1 + \dots + x_n v_n = \sum_{i=1}^{n} x_i v_i$

and hence $\underline{x}' = P^{-1} \underline{x}$ has

$$x_1' v_1' + \dots + x_n' v_n' = \sum_{i=1}^{n} x_i' v_i' = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p^{-1}_{ij} x_j \right) \left(\sum_{k=1}^{n} p_{ki} v_k \right)$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \underbrace{\underbrace{\left(\sum_{i=1}^{n} p_{ki} p^{-1}_{ij} \right)}_{(P \cdot P^{-1})_{kj}}}_{I_{kj}} x_j \right) v_k$$

$$= \sum_{k=1}^{n} x_k v_k = v \quad \blacksquare$$

EXAMPLE: For $V = \{ \text{solutions to } y''+y=0 \}$,

instead of fundamental solutions $y_1 = \cos(t)$, $y_2 = \sin(t)$ as $B = (y_1, y_2)$

one might learn $y''+y=0$ has characteristic equation $r^2+1=0$
$$(r-i)(r+i)=0$$

with roots $r_1, r_2 = \pm i$

and hence a different basis of solutions $B' = (e^{r_1 t}, e^{r_2 t})$
$$= (e^{it}, e^{-it}).$$

To check this, Euler's equation $e^{i\theta} = \cos\theta + i\sin\theta$ lets one write

$$e^{it} = \cos(t) + i\sin(t) = y_1 + iy_2$$
$$e^{-it} = \cos(t) - i\sin(t) = y_1 - iy_2$$

$$\rightsquigarrow P = {}_B\begin{array}{c|cc} & \overbrace{e^{it} \quad e^{-it}}^{B'} \\ \begin{array}{c}\cos(t)\\ \sin(t)\end{array} & \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \end{array}$$

base change matrix from B to B'

invertible!

Since $P^{-1} = \frac{1}{-2i}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$,

given $y = x_1\cos(t) + x_2\sin(t)$ with coordinates $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in B,

it will have $y = x_1' e^{it} + x_2' e^{-it}$ that is coordinates $\underline{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$ in B'

where $\underline{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = P^{-1}\underline{x} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x_1 - ix_2) \\ \frac{1}{2}(x_1 + ix_2) \end{bmatrix}.$

Chap. 4 Linear transformations

§4.1 The dimension formula

Recall $V \xrightarrow{T} W$ a map between F-vectorspaces V, W

is linear if $T(v_1+v_2) = T(v_1) + T(v_2)$

$T(cv) = cT(v)$

DEF'N: $\ker(T) := \{v \in V : T(v) = \underline{0}\} = T^{-1}(\underline{0})$, a subspace of V

kernel of T or nullspace of T, $\dim_F \ker(T) \overset{\text{DEF'N}}{=:} \text{nullity}(T)$

$\text{im}(T) := T(V) = \{w \in W : \exists v \in V \text{ with } T(v) = w\}$, a subspace of W,

image of T, $\dim_F \text{im}(T) \overset{\text{DEF'N}}{=:} \text{rank}(T)$

EXAMPLE: Consider $V \xrightarrow{T = \frac{d}{dx}} V$

$V = \left\{\begin{array}{l}\text{quadratic} \\ \text{polynomials} \\ f(x) = ax^2+bx+c \\ a,b,c \in \mathbb{R}\end{array}\right\}$

$f(x) \overset{T}{\longmapsto} T(f)(x) = f'(x) = \frac{df}{dx}$

$\dim(V) = 3$ since $\mathbb{R}^3 \xrightarrow{\varphi} V$ is an isomorphism.

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \longmapsto ax^2+bx+c$

What is $\ker(T)$?

$= \{f(x) = ax^2+bx+c : \frac{df}{dx} = 0\}$

$= \{f(x) = c\}$, $\text{nullity}(T) = 1$

What is $\text{im}(T)$?

$= \{\underset{\substack{\| \\ 2ax+b}}{f'(x)} : f(x) = ax^2+bx+c\} = \left\{\begin{array}{l}\text{linear} \\ \text{polynomials} \\ a'x+b\end{array}\right\}$, $\text{rank}(T) = 2$

THEOREM (The dimension formula) For any linear transformation,
(4.1.6)

$$V \xrightarrow{T} W, \text{ one has } \underset{\dim \ker(T)}{\underset{\parallel}{\text{nullity}(T)}} + \underset{\dim \text{im}(T)}{\underset{\parallel}{\text{rank}(T)}} = \dim V.$$

proof: We'll show something stronger:

similar in spirit to EXERCISE 7.10.2

given any basis $\{u_1, u_2, \ldots, u_k\}$ for $\ker(T)$,

and any basis $\{w_1, w_2, \ldots, w_r\}$ for $\text{im}(T)$,

if we pick any pre-images $\{v_1, \ldots, v_r\}$ with $T(v_1)=w_1, \ldots, T(v_r)=w_r$,

then $\{u_1, u_2, \ldots, u_k, v_1, \ldots, v_r\}$ gives a basis for V

(so $\dim V = k+r$
$= \dim \ker(T) + \dim \text{im}(T)$).

$\boxed{\text{They}}$ span V because for any $v \in V$,

one can write

$$\begin{aligned} T(v) &\in \text{im}(T) \\ \parallel & \\ c_1 w_1 + \cdots + c_r w_r &= c_1 T(v_1) + \cdots + c_r T(v_r) \\ &= T(c_1 v_1 + \cdots + c_r v_r) \end{aligned}$$

so $T\big(v - (c_1 v_1 + \cdots + c_r v_r)\big) = \underline{0}$

i.e. $v - (c_1 v_1 + \cdots + c_r v_r) \in \ker T$

$\Rightarrow \quad v - (c_1 v_1 + \cdots + c_r v_r) = d_1 u_1 + \cdots + d_k u_k$

$\Rightarrow \quad v = d_1 u_1 + \cdots + d_k u_k + c_1 v_1 + \cdots + c_r v_r$, as desired.

They are independent because if $d_1 u_1 + \cdots + d_k u_k + c_1 v_1 + \cdots + c_r v_r = \underline{0}$

then applying T shows $d_1 T(u_1) + \cdots + d_k T(u_k) + c_1 T(v_1) + \cdots + c_r T(v_r) = T(\underline{0}) = \underline{0}$

i.e. $c_1 T(v_1) + \cdots + c_r T(v_r) = \underline{0}$

$\Rightarrow c_1 = \cdots = c_r = 0$

But then $d_1 u_1 + \cdots + d_k u_k = \underline{0}$, so $d_1 = \cdots = d_k = 0$ also ∎

EXAMPLES:

(1) $V \xrightarrow{T=\frac{d}{dx}} V$ had $\text{nullity}(T) + \text{rank}(T) = \dim(V)$

$V = \{ax^2+bx+c : a,b,c \in \mathbb{R}\}$

$$\underset{1}{\text{nullity}(T)} + \underset{2}{\text{rank}(T)} = \underset{3}{\dim(V)}$$

12/5/2018 > (2) Sums and direct sums (part of § 3.6)

V_1+V_2 $\quad$ $V_1 \oplus V_2$

Given two vector spaces V_1, V_2 their direct sum $V_1 \oplus V_2 = V_1 \times V_2$ with componentwise $+$ and scaling

is another F-vector space, with $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$

because one can get a basis for $V_1 \oplus V_2$ by

~~concatenating~~ a basis $\{a_1, a_2, \ldots, a_{\dim V_1}\}$ for V_1

and a basis $\{b_1, b_2, \ldots, b_{\dim V_2}\}$ for V_2

and concatenating $\{(a_1, \underline{0}), \ldots, (a_{\dim V_1}, \underline{0}), (\underline{0}, b_1), \ldots, (\underline{0}, b_{\dim V_2})\}$

to get a basis for $V_1 \oplus V_2$

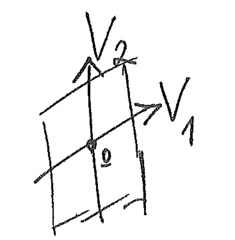
On the other hand, given two subspaces $V_1, V_2 \subset V$

one can form a new subspace $V_1 + V_2 := \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\} \subset V$

called their sum.

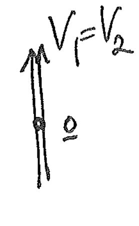
Q: What is $\dim(V_1+V_2)$?? It depends on $\dim(V_1)$, $\dim(V_2)$ and... how V_1, V_2 intersect

e.g. $V = \mathbb{R}^3$



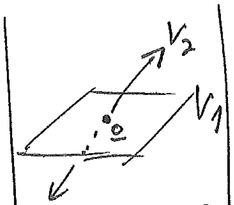
$V_1+V_2 =$ a plane

$\dim(V_1+V_2) = 2$



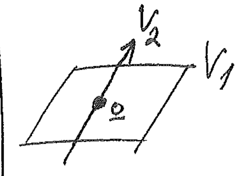
$V_1+V_2 = V_1 = V_2$

$\dim(V_1+V_2) = 1$



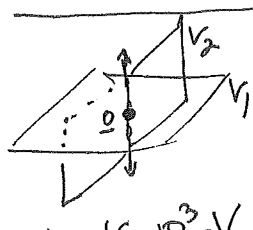
$V_1+V_2 = V = \mathbb{R}^3$

$\dim(V_1+V_2) = 3$



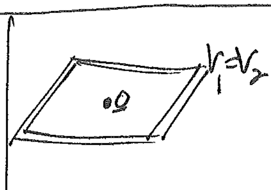
$V_1+V_2 = V_1$

$\dim(V_1+V_2) = 2$



$V_1+V_2 = \mathbb{R}^3 = V$

$\dim(V_1+V_2) = 3$



$V_1+V_2 = V_1 = V_2$

$\dim(V_1+V_2) = 2$