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EXAMPLES:

(1) $V \xrightarrow{T = \frac{d}{dx}} V$ had $\text{nullity}(T) + \text{rank}(T) = \dim(V)$

$$V = \{ax^2+bx+c : a,b,c \in \mathbb{R}\}, \qquad \underbrace{\text{nullity}(T)}_{=1} + \underbrace{\text{rank}(T)}_{=2} = \underbrace{\dim(V)}_{=3}$$

12/5/2018 → (2) Sums and direct sums (part of § 3.6)

$V_1 + V_2$ $\qquad$ $V_1 \oplus V_2$

Given two vector spaces V_1, V_2 their direct sum $V_1 \oplus V_2 = V_1 \times V_2$ with componentwise + and scaling

is another F-vector space, with $\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2)$

because one can get a basis for $V_1 \oplus V_2$ by taking

a basis $\{a_1, a_2, \ldots, a_{\dim V_1}\}$ for V_1

and a basis $\{b_1, b_2, \ldots, b_{\dim V_2}\}$ for V_2

and concatenating $\{(a_1, \underline{0}), \ldots, (a_{\dim V_1}, \underline{0}), (\underline{0}, b_1), \ldots, (\underline{0}, b_{\dim V_2})\}$

to get a basis for $V_1 \oplus V_2$

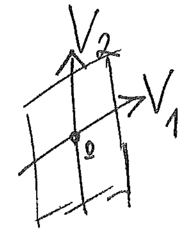
On the other hand, given two subspaces $V_1, V_2 \subset V$

one can form a new subspace $V_1 + V_2 := \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\} \subset V$

called their sum.

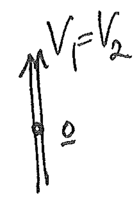
Q: What is $\dim(V_1 + V_2)$? It depends on $\dim(V_1)$, $\dim(V_2)$ and... how V_1, V_2 intersect

e.g. $V = \mathbb{R}^3$



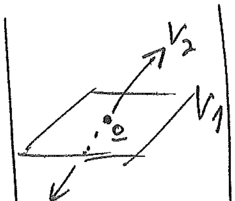
$V_1 + V_2 =$ a plane

$\dim(V_1 + V_2) = 2$



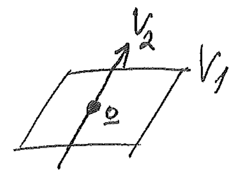
$V_1 + V_2 = V_1 = V_2$

$\dim(V_1 + V_2) = 1$



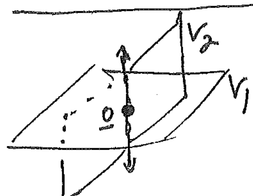
$V_1 + V_2 = V = \mathbb{R}^3$

$\dim(V_1 + V_2) = 3$



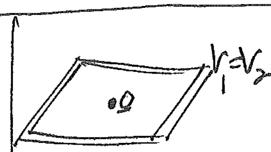
$V_1 + V_2 = V_1$

$\dim(V_1 + V_2) = 2$



$V_1 + V_2 = \mathbb{R}^3 = V$

$\dim(V_1 + V_2) = 3$



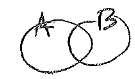
$V_1 + V_2 = V_1 = V_2$

$\dim(V_1 + V_2) = 2$

PROPOSITION: (3.6.6) Given V finite-dimensional and V_1, V_2 subspaces in V,

$$\dim(V_1+V_2)+\dim(V_1\cap V_2) = \dim V_1 + \dim V_2$$

reminiscent of $|A\cup B|+|A\cap B| = |A|+|B|$



e.g. check this in all of the previous examples

proof: There is always a linear transformation

$$V_1\oplus V_2 \xrightarrow{T} V_1+V_2 \ (\subset V)$$

$$(v_1, v_2) \longmapsto T((v_1,v_2)) = v_1+v_2$$

which is clearly surjective,

having $\ker(T) = \{(v_1,v_2) : v_1+v_2=0,$ i.e. $v_2=-v_1\}$

$= \{(v,-v) : v\in V_1\cap V_2\}$, which is isomorphic to $V_1\cap V_2$

via the map $\{(v,-v): v\in V_1\cap V_2\} \longrightarrow V_1\cap V_2$

$$(v,-v) \longmapsto v$$
$$(u,-u) \longleftarrow u.$$

Hence the dimension formula gives

$$\begin{array}{ccccc} \text{rank}(T) & + & \text{nullity}(T) & = & \dim(V_1\oplus V_2) \\ \| & & \| & & \| \\ \dim \operatorname{im}(T) & & \dim\ker(T) & & \\ \| & & \| & & \\ \dim(V_1+V_2) & + & \dim(V_1\cap V_2) & = & \dim(V_1)+\dim(V_2) \end{array} \quad \blacksquare$$

COROLLARY: In the setting of the PROPOSITION,

$V_1\oplus V_2 \to V$ is an isomorphism $\iff$ • $V_1+V_2=V$

$(v_1,v_2)\mapsto v_1+v_2$ • $V_1\cap V_2=\{0\}$

(reminiscent of $H,K<G$ have $H\times K \xrightarrow{\mu} G$ an isomorphism of groups

$(h,k)\mapsto hk$

$\iff$ • $HK=G$
• $H\cap K=\{1\}$
• H,K commute)

§ 4.2 Matrices & linear transformations

This is often the right way to think about a matrix.

LEMMA (4.2.1): Every matrix $A \in F^{m\times n}$ gives a linear transformation

$$F^n \xrightarrow{T} F^m$$
$$\underline{x} \longmapsto T(\underline{x}) = A\underline{x}$$

and conversely, every linear transformation $F^n \xrightarrow{T} F^m$ arises this way as $T(\underline{x}) = A\underline{x}$ for $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ where j^{th} column $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = T(e_j) = T\left(\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$, $\begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} = \sum_{i=1}^{n} a_{ij} e_i$

i.e. $A = \begin{bmatrix} | & & | \\ T(v_1) & \cdots & T(v_n) \\ | & & | \end{bmatrix}$

proof: Given $A \in F^{m\times n}$, certainly $T(\underline{x}) := A\underline{x}$ is linear, since

$$T(c\underline{x} + d\underline{y}) = A(c\underline{x} + d\underline{y}) = c\,A\underline{x} + d\,A\underline{y} = cT(\underline{x}) + dT(\underline{y}).$$

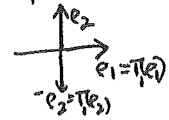
Conversely, given $F^n \xrightarrow{T} F^m$ linear, if we define A as above via $T(e_j) = \sum_{i=1}^{n} a_{ij} e_i$, then

$$T(\underline{x}) = T\Big(\sum_{j=1}^{n} x_j e_j\Big) = \sum_{j=1}^{n} x_j T(e_j)$$
$$= \sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij} e_i$$
$$= \sum_{i=1}^{n} \underbrace{\Big(\sum_{j=1}^{n} a_{ij} x_j\Big)}_{(A\underline{x})_i} e_i = A\underline{x} \quad \blacksquare$$

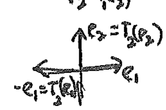
EXAMPLES:

① What are the matrices corresponding to these linear maps $\mathbb{R}^2 \xrightarrow{T_i} \mathbb{R}^2$?:

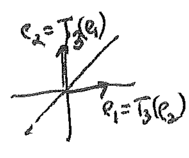
T_1 = reflection across x-axis $\rightsquigarrow A_1 = \begin{array}{c|cc} & e_1 & e_2 \\ e_1 & 1 & 0 \\ e_2 & 0 & -1 \end{array}$



T_2 = reflection across y-axis $\rightsquigarrow A_2 = \begin{array}{c|cc} & e_1 & e_2 \\ e_1 & -1 & 0 \\ e_2 & 0 & 1 \end{array}$



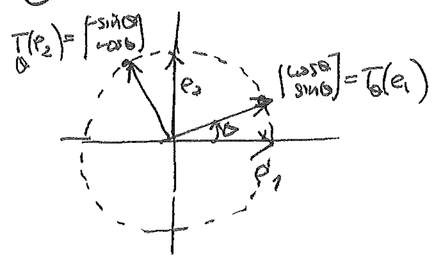
T_3 = reflection across x=y line $\rightsquigarrow A_3 = \begin{array}{c|cc} & e_1 & e_2 \\ e_1 & 0 & 1 \\ e_2 & 1 & 0 \end{array}$



(141)

(2) The map $\mathbb{R}^2 \xrightarrow{T_\theta} \mathbb{R}^2$ that rotates θ counterclockwise about 0

has matrix
$$A_\theta = \begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & \cos\theta & -\sin\theta \\ e_2 & \sin\theta & \cos\theta \end{array}$$



and since $T_{\theta_1} \circ T_{\theta_2} = T_{\theta_1 + \theta_2}$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "T_{\theta_2}"] \arrow[rr, bend right, "T_{\theta_1} \circ T_{\theta_2} = T_{\theta_1+\theta_2}"'] & \mathbb{R}^2 \arrow[r, "T_{\theta_1}"] & \mathbb{R}^2 \end{tikzcd} \Rightarrow \begin{tikzcd} \mathbb{R}^2 \arrow[r, "A_{\theta_2}"] \arrow[rr, bend right] & \mathbb{R}^2 \arrow[r, "A_{\theta_1}"] & \mathbb{R}^2 \end{tikzcd}$$

$$A_{\theta_1} A_{\theta_2} = A_{\theta_1 + \theta_2}$$

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \qquad \begin{bmatrix} \cos(\theta_1+\theta_2) & \sim \\ \sin(\theta_1+\theta_2) & \sim \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 & \sim \\ \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2 & \sim \end{bmatrix}$$

equality gives cos, sin addition formulas!

12/7/2018

~~What about~~ $V \xrightarrow{T} W$ when V, W aren't F^n, F^m?

If they're finite dimensional, pick bases for each of V & W and then one can write down a matrix for T.

PROPOSITION (4.2.5): Once you've picked a basis $\mathbb{B} = (v_1, \ldots, v_n)$ for $V \cong F^n$, $\quad \sum_{j=1}^n x_j v_j \xleftarrow{\psi_B} x$

and a basis $\mathbb{C} = (w_1, \ldots, w_m)$ for $W \cong F^m$, $\quad \sum_{i=1}^m y_i w_i \xleftarrow{\psi_C} y$

then the matrix $A = (a_{ij}) \in F^{m \times n}$

defined by $T(v_j) = \sum_{i=1}^m a_{ij} w_i$ makes this diagram commute:

$$\begin{tikzcd} F^n \arrow[r, "A"] \arrow[d, "\psi_{\mathbb{B}}"'] & F^m \arrow[d, "\psi_{\mathbb{C}}"] \\ V \arrow[r, "T"] & W \end{tikzcd} \qquad \begin{tikzcd} x \arrow[d, "\psi_B"'] & Ax = \psi_C^{-1}(T(\psi_B(x))) \\ v \arrow[r, mapsto] & T(v) \arrow[u, "\psi_{\mathbb{C}}^{-1}"'] \end{tikzcd}$$

$$v = \psi_B(x) \qquad T(v) = T(\psi_B(x))$$

i.e. $T \circ \psi_{\mathbb{B}} = \psi_{\mathbb{C}} \circ A$

so $\psi_{\mathbb{C}}^{-1} \circ T \circ \psi_{\mathbb{B}} = A$

> One calls A the matrix expressing T in the bases $\mathbb{B}$ for V and $\mathbb{C}$ for W