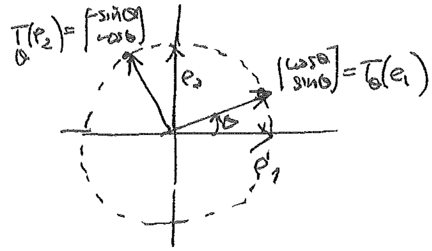
(2) The map $\mathbb{R}^2 \xrightarrow{T_\theta} \mathbb{R}^2$ that rotates θ counterclockwise about $\underline{0}$

has matrix $A_\theta = \begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & \cos\theta & -\sin\theta \\ e_2 & \sin\theta & \cos\theta \end{array}$



and since $T_{\theta_1} \circ T_{\theta_2} = T_{\theta_1+\theta_2}$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "T_{\theta_2}"] \arrow[rr, bend right, "T_{\theta_1}\circ T_{\theta_2} = T_{\theta_1+\theta_2}"'] & \mathbb{R}^2 \arrow[r, "T_{\theta_1}"] & \mathbb{R}^2 \end{tikzcd} \Rightarrow \begin{tikzcd} \mathbb{R}^2 \arrow[r, "A_{\theta_2}"] \arrow[rr, bend right] & \mathbb{R}^2 \arrow[r, "A_{\theta_1}"] & \mathbb{R}^2 \end{tikzcd}$$

$$A_{\theta_1} A_{\theta_2} = A_{\theta_1+\theta_2}$$

$$\begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix}\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \qquad \begin{bmatrix} \cos(\theta_1+\theta_2) & \sim \\ \sin(\theta_1+\theta_2) & \sim \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & \sim \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & \sim \end{bmatrix}$$

equality gives cos, sin addition formulas!

12/7/2018

~~What about~~ $V \xrightarrow{T} W$ when V, W aren't F^n, F^m?

If they're finite dimensional, pick bases for each of V & W and then one can write down a matrix for T.

PROPOSITION (4.2.5): Once you've picked a basis $\mathbb{B} = (v_1, \ldots, v_n)$ for $V \cong F^n$ $\quad \sum_{j=1}^{n} x_j v_j \overset{\psi_B}{\longleftarrow\!\shortmid} \underline{x}$

and a basis $\mathbb{C} = (w_1, \ldots, w_m)$ for $W \cong F^m$ $\quad \sum_{i=1}^{m} y_i w_i \overset{\psi_C}{\longleftarrow\!\shortmid} \underline{y}$

then the matrix $A = (a_{ij}) \in F^{m\times n}$

defined by $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$ makes this diagram commute:

$$\begin{tikzcd} F^n \arrow[r, "A"] \arrow[d, "\psi_{\mathbb{B}}"'] & F^m \arrow[d, "\psi_{\mathbb{C}}"] \\ V \arrow[r, "T"] & W \end{tikzcd} \qquad \begin{tikzcd} \underline{x} \arrow[d, "\psi_B"'] & A\underline{x} = \psi_C^{-1}(T(\psi_B(x))) \\ v \arrow[r, mapsto] & T(v) \arrow[u, "\psi_{\mathbb{C}}^{-1}"'] \end{tikzcd}$$

$v = \psi_B(x)$, $\quad T(v) = T(\psi_B(x))$

i.e. $T \circ \psi_{\mathbb{B}} = \psi_{\mathbb{C}} \circ A$

so $\psi_{\mathbb{C}}^{-1} \circ T \circ \psi_{\mathbb{B}} = A$

One calls A the <u>matrix expressing</u> T <u>in the bases</u> <u>$\mathbb{B}$ for V</u> and <u>$\mathbb{C}$ for W</u>

proof: Let's check $(T \circ \psi_{\mathbb{B}})(\underline{x}) \overset{?}{=} (\psi_{\mathbb{C}} \circ A)(\underline{x}) \quad \forall \underline{x} \in F^n$

$$
\begin{aligned}
T(\psi_{\mathbb{B}}(\underline{x})) &= T\left(\sum_{j=1}^{n} x_j v_j\right) = \sum_{j=1}^{n} x_j T(v_j) = \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{m} a_{ij} w_i\right) \\
&= \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} a_{ij} x_j\right)}_{(A\underline{x})_i} w_i = \psi_{\mathbb{C}}(A\underline{x}) \\
&= (\psi_{\mathbb{C}} \circ A)(\underline{x}) \checkmark \quad \blacksquare
\end{aligned}
$$

And what happens if we change bases in V or in W;

how will the matrix A for T change?

Easy — by invertible matrices Q, P multiplying A on left, right
(row ops!) (column ops!)

PROPOSITION 4.2.13: If the linear transformation $V \xrightarrow{T} W$ has matrix $A \in F^{m \times n}$ w.r.t. bases $\mathbb{B}$ for V, $\mathbb{C}$ for W

and we pick new bases $\mathbb{B}' = (v_1', \ldots, v_n')$ for V
$\mathbb{C}' = (w_1', \ldots, w_m')$ for W

with change-of-basis matrices $P \in F^{n \times n}$, so $\underline{x} \overset{\psi_{\mathbb{B}}}{\longmapsto} v$ means $\underbrace{P^{-1}\underline{x}}_{\underline{x}'} \overset{\psi_{\mathbb{B}'}}{\longmapsto} v$
$Q \in F^{m \times m}$, $\underline{y} \overset{\psi_{\mathbb{C}}}{\longmapsto} w$ means $\underbrace{Q^{-1}\underline{y}}_{\underline{y}'} \overset{\psi_{\mathbb{C}'}}{\longmapsto} w$

then T has matrix $A' = Q^{-1}AP$ w.r.t. bases $\mathbb{B}'$ for V, $\mathbb{C}'$ for W.

proof: We should have a commutative diagram for some matrix A'

$$
\begin{tikzcd}
P\underline{x}' & \underline{x} \arrow[dd, mapsto] & (\mathbb{B})\ F^n \arrow[r, "A"] \arrow[d, "\psi_{\mathbb{B}}"'] & F^m\ (\mathbb{C}) \arrow[d, "\psi_{\mathbb{C}}"] & \underline{y} \arrow[dd, mapsto] & Q\underline{y}' \\
 & & V \arrow[r, "T"] & W & & \\
\underline{x}' \arrow[uu, mapsto] & P^{-1}\underline{x} & (\mathbb{B}')\ F^n \arrow[r, "A'"] \arrow[u, "\psi_{\mathbb{B}'}"] & F^m\ (\mathbb{C}') \arrow[u, "\psi_{\mathbb{C}'}"'] & Q^{-1}\underline{y} & \underline{y}' \arrow[uu, mapsto]
\end{tikzcd}
$$

and so we can follow $\underline{x}'$ around:

$$
\begin{tikzcd}
P\underline{x}' \arrow[r, mapsto, "A"] & AP\underline{x}' = \underline{y} \arrow[d, mapsto] \\
\underline{x}' \arrow[u, mapsto] & Q^{-1}\underline{y} = Q^{-1}AP\underline{x}'
\end{tikzcd}
$$

Thus $A' = Q^{-1}AP$ $\blacksquare$

This has some interesting consequences.

THEOREM (4.2.10): (a) Every linear transformation $V \xrightarrow{T} W$ with $\dim(V)=n$, $\dim(W)=m$ rank r $\iff$ there is has a choice of bases $\mathbb{B}$ for V, $\mathbb{C}$ for W

so that its matrix looks

$$A = \begin{array}{c} \\ r\{ \\ m-r\{ \end{array} \begin{array}{cc} \overbrace{\quad}^{r} & \overbrace{\quad}^{n-r} \\ \left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$$

(b) For every matrix $A \in F^{m\times n}$, one has $\mathrm{rank}(A)=r$

$$\iff \exists\ P \in GL_n(F),\ Q \in GL_m(F) \text{ with } Q^{-1}AP = \begin{array}{c} \\ r\{ \\ m-r\{ \end{array} \begin{array}{cc} \overbrace{\quad}^{r} & \overbrace{\quad}^{n-r} \\ \left[\begin{array}{c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$$

<u>proof</u>: (a) Let $r = \mathrm{rank}(T) = \dim \mathrm{im}(T)$, so $\exists$ a basis $w_1, \dots, w_r$ for $\mathrm{im}(T) \subset W$.

Extend it to ~~an~~ ordered basis $(w_1, \dots, w_r, w_{r+1}, \dots, w_n) =: \mathbb{C}$ for W,

and choose lifts $(v_1, \dots, v_r) \in V$ with $T(v_i) = w_i$

along with a basis $u_1, \dots, u_k$ for $\ker(T) \subset V$, $k = n-r$

so we've seen $(v_1, \dots, v_r, u_1, \dots, u_{n-r}) =: \mathbb{B}$ is a basis for V.

But then the matrix A for T in bases $\mathbb{B}, \mathbb{C}$ looks like

$$A = \begin{array}{c} \\ \mathrm{im}(T)\left\{\begin{matrix} w_1 \\ \vdots \\ w_r \end{matrix}\right. \\ \begin{matrix} w_{r+1} \\ \vdots \\ w_n \end{matrix} \end{array} \begin{array}{cc} v_1 \cdots v_r & \overbrace{u_1, \dots, u_{n-r}}^{\ker(T)} \\ \left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$$

$$\begin{tikzcd} F^n \arrow[r, "A"] \arrow[d, "\psi_{\mathbb{B}}"'] & F^m \arrow[d, "\psi_{\mathbb{C}}"] \\ V \arrow[r, "T"] & W \end{tikzcd}$$

(b) Let $F^n \xrightarrow{A} F^m$, $\underline{x} \longmapsto A\underline{x}$ play the role of $V \xrightarrow{T} W$ in part (a),

and make the change of basis to $\mathbb{B}$ for $V = F^n$, $\mathbb{C}$ for $W = F^m$ as in (a),

then one gets

$$\begin{tikzcd} F^n \arrow[r, "A' = Q^{-1}AP"] \arrow[d, "\psi_{\mathbb{B}}"'] & F^m \arrow[d, "\psi_{\mathbb{C}}"] \\ F^n \arrow[r, "A"] & F^m \end{tikzcd}$$

with $A' = \left[\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right]$, $A' = Q^{-1}AP$ for some $P \in GL_n(F)$, $Q \in GL_m(F)$ ∎

Another consequence:

THEOREM (4.2.14) $\quad \operatorname{rank}(A) = \operatorname{rank}(A^T) \quad \forall A \in F^{m\times n}$

proof: $\operatorname{rank}(A) = r \iff \exists \begin{matrix} P \in GL_n(F) \\ Q \in GL_m(F) \end{matrix}$ with $Q^{-1}AP = \begin{array}{c} \\ r\{ \\ m-r\{ \end{array} \!\!\begin{array}{c} \overbrace{}^{r} \quad \overbrace{}^{n-r} \\ \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$

i.e. $(Q^{-1}AP)^T = \begin{array}{c} \\ r\{ \\ m-r\{ \end{array} \!\!\begin{array}{c} \overbrace{}^{r} \quad \overbrace{}^{m-r} \\ \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$

$\quad\quad \| $

$P^T A^T (Q^{-1})^T$

$\iff \exists \begin{matrix} \hat{P} \in GL_m(F) \\ \hat{Q} \in GL_n(F) \end{matrix}$ with $\hat{Q}^{-1} A^T \hat{P} = \begin{array}{c} \\ r\{ \\ n-r\{ \end{array} \!\!\begin{array}{c} \overbrace{}^{r} \quad \overbrace{}^{m-r} \\ \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] \end{array}$

let $\hat{Q}^{-1} := P^T$
and $\hat{P} := (Q^{-1})^T$

$\iff \operatorname{rank}(A^T) = r$ ■

MORAL: When working over fields F, all linear transformations $V \xrightarrow{T} W$ of same rank can be made to look the same by changes-of-bases in V, W. (We'll see it's different working over $\mathbb{Z}$ later on!)
= Chap. 14

---

§4.3 Linear operators

DEF'N: When $V \xrightarrow{T} V$ is linear, you call it a linear operator

In that case, it makes sense to pick one basis $\mathbb{B} = (v_1, \ldots, v_n)$ for V (if it's finite-dimensional)) and use it for both copies of V, to write down the matrix A for T w.r.t. $\mathbb{B}$;

$$\begin{tikzcd} \underline{x} \arrow[r, mapsto] & A\underline{x} \\ F^n \arrow[r, "A"] \arrow[d, "\varphi_{\mathbb{B}}"'] & F^n \arrow[d, "\varphi_{\mathbb{B}}"] \\ V \arrow[r, "T"] & V \end{tikzcd}$$

$\mathbb{B} = (v_1, \ldots, v_n) \qquad \mathbb{B} = (v_1, \ldots, v_n)$

namely $A = (a_{ij})_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}} \in F^{n\times n}$

where $T(v_j) = \sum_{i=1}^{n} a_{ij} v_i$

$$A = \begin{array}{c} \\ v_1 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{array} \!\!\begin{array}{c} v_1 \cdots v_j \cdots v_n \\ \left[\begin{array}{ccc} & & \\ & a_{ij} & \\ & & \end{array}\right] \end{array} = \begin{bmatrix} | & & | \\ T(v_1) & \cdots & T(v_n) \\ | & & | \end{bmatrix}$$

(145) **PROPOSITION:** If $V \xrightarrow{T} W$ had matrix A w.r.t. basis $\mathbb{B} = (v_1, \ldots, v_n)$
(4.3.5)

$A = (a_{ij}) \in F^{n \times n}$

then w.r.t. a new basis $\mathbb{B}' = (v_1', \ldots, v_n')$

that has $v_j' = \sum_{i=1}^{n} p_{ij} v_i$ for $P = (p_{ij}) \in GL_n(F)$ as change-of-basis matrix,

T will have matrix $A' = P^{-1}AP$ i.e. conjugation of A by P. (similarity transformation)

12/10/2018 >

proof: Follows from what we did, but let's just do a sanity check.

We have $T(v_j) = \sum_{i=1}^{n} a_{ij} v_i$ for $j = 1, \ldots, n$

and also $v_j = \sum_{i=1}^{n} p^{-1}_{ij} v_i'$,

so
$$\begin{aligned} T(v_j') &= T\Big(\sum_{i=1}^{n} p_{ij} v_i\Big) = \sum_{i=1}^{n} p_{ij} T(v_i) \\ &= \sum_{i=1}^{n} p_{ij} \sum_{k=1}^{n} a_{ki} v_k \\ &= \sum_{i=1}^{n} p_{ij} \sum_{k=1}^{n} a_{ki} \sum_{l=1}^{n} p^{-1}_{lk} v_l' \\ &= \sum_{l=1}^{n} \underbrace{\Big(\sum_{k=1}^{n} \sum_{i=1}^{n} p^{-1}_{lk} a_{ki} p_{ij}\Big)}_{(P^{-1}AP)_{lj}} v_l' , \text{ as desired } \blacksquare \end{aligned}$$

So the idea of §4.4, 4.5, 4.6 is to figure out how to make particularly useful change-of-basis $\mathbb{B} \rightsquigarrow \mathbb{B}'$

so that $A \rightsquigarrow PAP^{-1}$ becomes particularly simple.