(111)

11/12/2018

THEOREM: Given a set map $S \xrightarrow{f} G$ a group, $s_i \longmapsto f(s_i)$

such that every relation $r \in R \subset F(S) \xrightarrow{\varphi} G$, $s_i \longmapsto f(s_i) = \varphi(s_i)$

has $\varphi(r) = 1$ (i.e. $R \subset \ker\varphi$)

then there is a unique induced homomorphism

$$\langle S \mid R \rangle = F(S) / \langle R \rangle_{normal} \xrightarrow{\bar{\varphi}} G$$

sending $\bar{s}_i \longmapsto \bar{\varphi}(\bar{s}_i) = f(s_i) = \varphi(s_i)$

proof: This is really just part of a more general universal property for quotient groups: (PROP 7.10.13)

Given $G' \xrightarrow{\varphi} G$ a group homomorphism and normal subgroup $K \lhd G'$ such that $K < \ker\varphi$, there is a unique induced homomorphism

$$G'/K \xrightarrow{\bar{\varphi}} G$$

(given by $g'K \longmapsto \bar{\varphi}(g'K) = \varphi(g')$),

such that this diagram commutes,

$$\begin{tikzcd} G'/K \arrow[r, "\bar{\varphi}"] & G \\ G' \arrow[u, "\pi"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

meaning $\varphi = \bar{\varphi} \circ \pi$

proof: This thing just proves itself - try it! ∎

Then the THM follows by taking $G' = F(S)$, $K \lhd$, $K = \langle R \rangle_{normal}$.

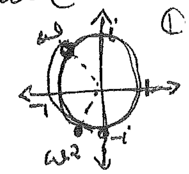
COROLLARY: $\langle s, r \mid s^2, r^n, srsr \rangle \cong D_n$

proof: The map $s \xrightarrow{f} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $r \longmapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\theta = \frac{2\pi}{n}$ induces $F(\{s,r\}) \xrightarrow{\varphi} D_n$ that has $s^2, r^n, srsr \in \ker\varphi$, so it induces a homomorphism $\langle s, r \mid s^2, r^n, srsr \rangle \xrightarrow{\bar{\varphi}} D_n$, which surjects since $D_n = \left\langle \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right\rangle$.

But then $\bar{\varphi}$ must be an isomorphism since the LHS has $\leq 2n$ elements, and $|D_n| = 2n$ ∎

EXAMPLE (from §7.8)

Inside $GL_2(\mathbb{C})$, consider the group $G := \left\langle \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{X=}, \underbrace{\begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}}_{Y=} \right\rangle$ where $\omega = e^{2\pi i/3}$



$X^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\qquad Y^2 = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega \end{bmatrix}$

$X^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\qquad \boxed{Y^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

$\boxed{X^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$

$$XYX^{-1} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega^2 \\ \omega & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega \end{bmatrix} = Y^{-1} = Y^2$$

i.e. $\boxed{XYX^{-1} = Y^2}$

or $\boxed{XY = Y^2X}$

or $\boxed{XYX^{-1}Y^{-2} = 1}$

This already shows $|G| \le \underset{\overset{\parallel}{12}}{3 \cdot 4}$ elements, namely $\{1, X, X^2, X^3,$
$Y, XY, X^2Y, X^3Y,$
$Y^2, XY^2, X^2Y^2, X^3Y^2\}$
$= \{X^iY^j : 0 \le i \le 3, 0 \le j \le 2\}$

but then G has exactly 12 elements, since it contains at least

$$\left\{ \pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm\begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}, \pm\begin{bmatrix} \omega^2 & 0 \\ 0 & \omega \end{bmatrix}, \pm\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 0 & \omega \\ -\omega^2 & 0 \end{bmatrix}, \pm\begin{bmatrix} 0 & \omega^2 \\ -\omega & 0 \end{bmatrix} \right\} \subset GL_2(\mathbb{C})$$

This also shows $\langle S | R \rangle = \langle x, y \mid x^4, y^3, xyx^{-1}y^{-2} \rangle \cong G$

via the isomorphism
$x \longmapsto X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$y \longmapsto Y = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix}$

REMARK:

Working with groups $G = \langle S | R \rangle$ presented by generators S, relations R can be tricky, because sometimes the relations have extra consequences that one might not guess.

e.g. $F(\{a\}) \cong \mathbb{Z}^+$ (no surprise)

$$G_1 = \langle \underset{S}{\{a\}} | \underset{R}{\{a^n\}} \rangle = F(\{a\}) / \langle a^n \rangle_{\text{normal}} = \{1, a, a^2, \ldots, a^{n-1}\} \cong (\mathbb{Z}/n\mathbb{Z})^+$$

but $G_2 = \langle \{a\} | \{a^{10}, a^{27}\} \rangle = \{1\}$ since in G_2 one has

$$a = a^1 = a^{3 \cdot 27 - 8 \cdot 10} = (a^{-10})^8 (a^{27})^3 = 1^8 \cdot 1^3 = 1$$

<u>THM</u> (Novikov-Boone, 1955, 1988) There is <u>no algorithm</u> taking S, R as input and two words $\underline{u}, \underline{v} \in S \sqcup S^{-1}$ that can always decide whether $\underline{u} = \underline{v}$ in $\langle S | R \rangle$

The <u>Todd-Coxeter algorithm</u> presented in Artin §7.11 is a cool algorithm that can answer such questions in $\langle S | R \rangle$, <u>if we know ahead of time that $|\langle S | R \rangle|$ is finite</u>. Otherwise, Todd-Coxeter may <u>not terminate</u> for some inputs.

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§7.8 Groups of order 12

Here they are:

THEOREM: (THM 7.8.1) A group G with $|G| = 12$ is either isomorphic to

abelian ↑

- $(\mathbb{Z}/4\mathbb{Z})^+ \times (\mathbb{Z}/3\mathbb{Z})^+$ (Sun Ze $\cong (\mathbb{Z}/12\mathbb{Z})^+$ cyclic)
- $(\mathbb{Z}/2\mathbb{Z})^+ \times (\mathbb{Z}/2\mathbb{Z})^+ \times (\mathbb{Z}/3\mathbb{Z})^+$ $(\cong (\mathbb{Z}/2\mathbb{Z})^+ \times (\mathbb{Z}/6\mathbb{Z})^+)$

non-abelian ↓

- A_4 $(< S_4)$ alternating group
- D_6 dihedral group
- $\left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix} \right\rangle < GL_2(\mathbb{C})$, iso. to $\langle x, y | x^4 = 1 = y^3, xy = y^2 x \rangle$