REMARK:

Working with groups $G = \langle S \mid R \rangle$ presented by generators S, relations R can be tricky, because sometimes the relations have extra consequences that one might not guess.

e.g. $F(\{a\}) \cong \mathbb{Z}^+$ (no surprise)

$G_1 = \langle \underset{S}{\underset{\parallel}{\{a\}}} \mid \underset{R}{\underset{\parallel}{\{a^n\}}} \rangle = F(\{a\}) / \langle a^n \rangle_{\text{normal}} = \{1, a, a^2, \ldots, a^{n-1}\} \cong (\mathbb{Z}/n\mathbb{Z})^+$

but $G_2 = \langle \{a\} \mid \{a^{10}, a^{27}\} \rangle = \{1\}$ since in G_2 one has

$a = a^1 = a^{3 \cdot 27 - 8 \cdot 10} = (a^{-10})^8 (a^{27})^3 = 1^8 \cdot 1^3 = 1$

THM (Novikov-Boone) (1955, 1958) There is no algorithm taking S, R as input and two words $\underline{u}, \underline{v} \in S \sqcup S^{-1}$ that can always decide whether $\underline{u} = \underline{v}$ in $\langle S \mid R \rangle$

The Todd-Coxeter algorithm presented in Artin §7.11 is a cool algorithm that can answer such questions in $\langle S \mid R \rangle$, if we know ahead of time that $|\langle S \mid R \rangle|$ is finite. Otherwise, Todd-Coxeter may not terminate for some inputs.

11/14/2018

§7.8 Groups of order 12

Here they are:

THEOREM: (THM 7.8.1) A group G with $|G| = 12$ is either isomorphic to

abelian ↑

- $(\mathbb{Z}/4\mathbb{Z})^+ \times (\mathbb{Z}/3\mathbb{Z})^+$ (Sun Ze: $\cong (\mathbb{Z}/12\mathbb{Z})^+$ cyclic)
- $(\mathbb{Z}/2\mathbb{Z})^+ \times (\mathbb{Z}/2\mathbb{Z})^+ \times (\mathbb{Z}/3\mathbb{Z})^+$ ($\cong (\mathbb{Z}/2\mathbb{Z})^+ \times (\mathbb{Z}/6\mathbb{Z})^+$)

non-abelian ↓

- A_4 ($< S_4$) alternating group
- D_6 dihedral group
- $\left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix} \right\rangle < GL_2(\mathbb{C})$, iso. to $\langle x, y \mid x^4 = 1 = y^3, xy = y^2 x \rangle$

<u>proof:</u> Since $|G| = 12 = 2^2 \cdot 3^1$,

- If $H < G$ is a Sylow 2-subgroup, so $|H| = 4$ and $H \cong (\mathbb{Z}/4\mathbb{Z})^+$ or $\underbrace{(\mathbb{Z}/2\mathbb{Z})^+ \times (\mathbb{Z}/2\mathbb{Z})^+}_{\cong V_4}$
  then $s_2 \mid 3 \Rightarrow s_2 = 1$ or 3 ($s_2 \equiv 1 \bmod 2$ tells us nothing more)
- If $K < G$ is a Sylow 3-subgroup, so $|K| = 3$ and $K \cong (\mathbb{Z}/3\mathbb{Z})^+$
  then $s_3 \mid 4 \Rightarrow s_3 = 1$, $\cancel{2}$, or 4
  impossible, since $s_3 \equiv 1 \bmod 3$.

(Although he doesn't use it, Artin then explains why <u>at least one</u> of H or K is normal G, via a cardinality argument – let's skip it)

Since $|H \cap K|$ divides 3 and 4, $|H \cap K| = 1$ i.e. $H \cap K = \{1\}$

and the map $\begin{array}{ccc} H \times K & \xrightarrow{\mu} & G \\ (h, k) & \longmapsto & hk \end{array}$ must be <u>injective</u>, hence <u>bijective</u> since $|G| = 12 = 3 \cdot 4 = |H \times K|$

Thus $G = HK$ i.e. every $g \in G$ can be written uniquely as $g = hk$.

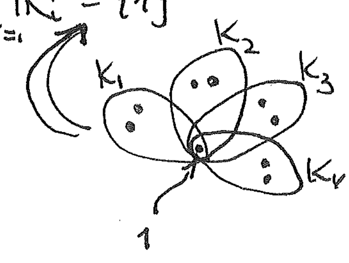
<u>CASE 1:</u> $H, K \triangleleft G$

Then we've seen this forces H, K to commute ($hk = kh \ \forall h \in H, k \in K$)

and μ is a group isomorphism: $G \cong H \times K$
$\cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
or
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

<u>CASE 2:</u> $K \ntriangleleft G$, i.e. $s_3 = 4$

Considering the action of G on $S = \{$ Sylow 3-subgroups $\overset{K}{\overset{\shortparallel}{K_1}}, K_2, K_3, K_4 \}$
via <u>conjugation</u> $g * K_i = gK_ig^{-1}$,
we know it is a <u>transitive action</u> (Sylow's 2nd), and $4 = s_3 = [G : N_G(K_i)] = \frac{12}{|N_G(K_i)|}$
$\Rightarrow |N_G(K_i)| = 3$. Since $K_i < N_G(K_i)$, this means $K_i = N_G(K_i)$
and so the kernel of the action $\bigcap_{i=1}^{4} N_G(K_i) = \bigcap_{i=1}^{4} K_i = \{1\}$



Thus we get an <u>injective</u> homomorphism
$G \xrightarrow{\varphi} S_{\{K_1, K_2, K_3, K_4\}} \cong S_4$
i.e. $G \cong \operatorname{im}\varphi < S_4$

We claim this forces $G \cong A_4$, because $|G| = 12 = \frac{|S_4|}{2}$:

LEMMA: For $n \geq 2$, the only subgroup $G < S_n$ with $|G| = \frac{n!}{2}$ (i.e. $[S_n : G] = 2$) is $G = A_n$.

proof: We've seen any such subgroup $G \lhd S_n$ (since any $\sigma \in S_n \setminus G$ has $\sigma G = S_n \setminus G = G\sigma$)

so one gets the canonical surjection

$$S_n \xrightarrow{\pi} S_n/G \cong \{\pm 1\} \quad \text{having } G = \ker(\pi).$$

But π must send every transposition $(ij) \in S_n$ to the same image $\pi((ij)) = \pm 1$, as they are all conjugate and $\{\pm 1\}$ is abelian:

$$(i'j') = p\,(ij)\,p^{-1} \text{ for some } p \in S_n$$

$$\Rightarrow \pi(i'j') = \pi(p)\,\pi(ij)\,\pi(p)^{-1} = \pi(p)\cdot\pi(p)^{-1}\cdot\pi(ij) = \pi(ij).$$

Since transpositions generate S_n, if all $\pi((ij)) = +1$, then $\pi(p) = +1 \ \forall p \in S_n$, so $\underset{G}{\ker(\pi)} = S_n$ contradiction to $|G| = \frac{|S_n|}{2}$.

If all $\pi((ij)) = -1$, then $\pi = \text{sign}$ and $G = \ker(\pi) = A_n$ ∎

CASE 3: $K \lhd G$, but $H \ntriangleleft G$. Name $K = \{1, y, y^2\} \cong (\mathbb{Z}/3\mathbb{Z})^+$

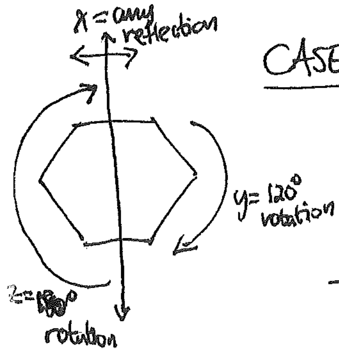
CASE 3a: $H \cong (\mathbb{Z}/4\mathbb{Z})^+$, say $H = \langle x \rangle = \{1, x, x^2, x^3\}$

Since $H \ntriangleleft G$, x and y cannot commute, so $xyx^{-1} \neq 1$, $\neq y$, $= y^2$

This means we have a surjective homomorphism

$$\langle x, y \mid x^4 = 1 = y^3,\ xyx^{-1} = y^2 \rangle \xrightarrow{\bar{\varphi}} G$$

and since both sides have cardinality 12, $\bar{\varphi}$ is an isomorphism.



CASE 3b: $H \cong (\mathbb{Z}/2\mathbb{Z})^+ \times (\mathbb{Z}/2\mathbb{Z})^+$, say $H = \langle x, z \rangle = \{1, x, z, \overset{zx}{\overset{\shortparallel}{xz}}\}$, $x^2 = z^2 = 1$

Since $H \ntriangleleft G$, some element of $H - \{1\}$ does not commute with y, and WLOG we can assume it is x, so $xyx^{-1} = y^2$.

Then if $zyz^{-1} = y^2$, we would have $xzy(xz)^{-1} = y$. So WLOG $zyz^{-1} = y$ i.e. z, y commute.

Thus $G' = \langle x, y, z \mid x^2 = z^2 = y^3 = 1,\ xy = y^2x,\ xz = zx,\ yz = zy \rangle \xrightarrow{\bar{\varphi}} G$ has a surjective homomorphism, and $|G'| \leq 2 \cdot 2 \cdot 3 = 12 = |G|$, if such a G with $|G| = 12$ exists.

But $G = D_6$ has such x, y, z in it – see picture at left. ∎