DEF'N: $GL_n(F) := \{A \in F^{n\times n} : A \text{ has an (2-sided) inverse in } F^{n\times n}\}$
$= \{A \in F^{n\times n} : \det A \neq 0 \text{ in } F\}$
i.e. $\det A \in F^\times$

and $SL_n(F) = \{A \in F^{n\times n} : \det A = 1\}$
$= \ker(\det)$ where $GL_n(F) \xrightarrow{\det} F^\times$

Whenever F is a finite field like $F = \mathbb{F}_p$ (and we'll see next semester there are finite fields $\mathbb{F}_{p^r}$ with $|\mathbb{F}_{p^r}| = p^r$ $\forall r \geq 1$)

then $GL_n(F)$, $SL_n(F)$ are finite groups that we haven't discussed yet.

EXAMPLE: $GL_2(\mathbb{F}_2) = SL_2(\mathbb{F}_2) = \left\{ \begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}, \begin{bmatrix}1&1\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\1&1\end{bmatrix}, \begin{bmatrix}0&1\\1&1\end{bmatrix}, \begin{bmatrix}1&1\\1&0\end{bmatrix} \right\}$

order: 1, 2, 2, 2, 3, 3

$\cong D_3 \cong S_3$

Digression on simple groups

Although classifying all finite groups G seems hopeless, in the late 1800's, Otto Hölder suggested a two-part program:

(1) Classify all the finite simple groups G, that is, those with no normal subgroup K having $1 \subsetneq K \underset{\neq}{\trianglelefteq} G$

(2) Understand all the ways to assemble a group G from a normal subgroup K and the quotient group G/K

Part (2) is still somewhat hard, but has motivated a lot of group theory, including group cohomology.

Part (1) turned out to be do-able in the 20^{th} century, but with a ton of tedious work...

EXAMPLES of simple groups

In fact, this is essentially the full classification of finite simple groups

① $(\mathbb{Z}/p\mathbb{Z})^+$ for primes p — easy to see simple
= cyclic groups

② A_n ($< S_n$) for $n \geq 5$ is simple — requires proof, maybe later
alternating group

(not for $n=4$ since we've seen $1 \subsetneq V_4 \triangleleft A_4$
$V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$

and for $n=3$, $A_3 \cong (\mathbb{Z}/3\mathbb{Z})^+$)

③ Not $GL_n(\mathbb{F}_{p^r})$, since $SL_n(\mathbb{F}_{p^r}) \triangleleft GL_n(\mathbb{F}_{p^r})$

and not $SL_n(\mathbb{F}_{p^r})$ since $Z(SL_n(\mathbb{F}_{p^r})) \triangleleft SL_n(\mathbb{F}_{p^r})$

$Z(SL_n(\mathbb{F}_{p^r})) = \{$scalar matrices with det 1$\} = \left\{ \begin{bmatrix} c & & 0 \\ & \ddots & \\ 0 & & c \end{bmatrix} : c^n = 1 \text{ in } \mathbb{F}_{p^r} \right\}$

but $PSL_n(\mathbb{F}_{p^r}) := SL_n(\mathbb{F}_{p^r}) / Z(SL_n(\mathbb{F}_{p^r}))$

is simple (except for $n=2$ with $p^r = 2$ or 3) — requires proof, not so hard

④ There are other infinite families matrix groups over finite fields, like $PSL_n(\mathbb{F}_{p^r})$, called finite groups of Lie type, which are simple.

⑤ Then there are 27 "sporadic" finite simple groups!
The largest, called the Monster has order $\approx 8 \times 10^{53}$!?