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§3.3 Vector spaces

We're used to them mostly over $\mathbb{R}$, like $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4$

but they make sense over any field F.

DEF'N: A vector space V over a field F ("scalars") is a set V with

- a composition law $V \times V \xrightarrow{+} V$, $(v_1, v_2) \longmapsto v_1 + v_2$, making $V^+ = (V, +)$ an abelian group, whose (additive) identity is called the zero vector $\underline{0}$

- and a scalar multiplication $F \times V \longrightarrow V$, $(c, v) \longmapsto cv$

satisfying these further rules

- $1 \cdot v = v \quad \forall v \in V$ (here $1 \in F$)
- $a(bv) = (ab)v \quad \forall a, b \in F,\ v \in V$
- two kinds of distributivity:

$$(a+b)v = av + bv \quad \forall a, b \in F,\ v \in V$$

$$a(v_1 + v_2) = av_1 + bv_2 \quad \forall a \in F,\ v_1, v_2 \in V$$

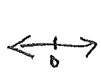
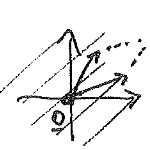
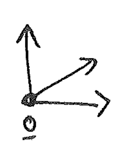
EXAMPLES:

① Most important example: $V = F^n := \left\{ n\text{-dimensional column vectors } v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} : a_i \in F \right\}$

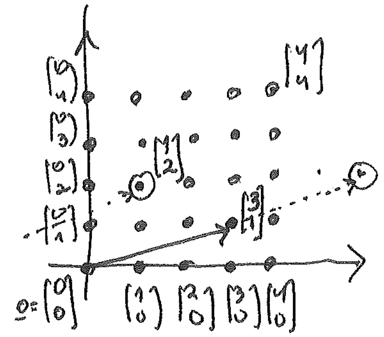
$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}, \qquad \underline{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$c \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}$$

e.g. $F = \mathbb{R}$

$V = \mathbb{R}^1$  | $V = \mathbb{R}^2$  | $V = \mathbb{R}^3$ 

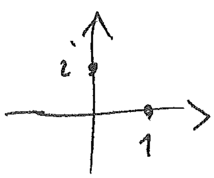
One can try to picture $V = (\mathbb{F}_5)^2$ over $F = \mathbb{F}_5$, but things become tricky due to "wrap-around":



$$2 \cdot \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}6\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$$

(2) $V = \mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ can be regarded as an $\mathbb{R}$-vector space (a vector space over $F = \mathbb{R}$)

$\mathbb{C}$



$c \in \mathbb{R}$

$c(a+bi) = ca + cbi$

$(a+bi) + (a'+b'i) = (a+a') + (b+b'i)$

It "looks" a lot like $V = \mathbb{R}^2$; let's make this precise.

DEF'N: A map $V \xrightarrow{\varphi} V'$ between two F-vector spaces V, V'
$v \longmapsto \varphi(v)$
is called linear if it respects the two pieces of vector-space structure:

- φ is an (abelian) group homomorphism $V^+ \to (V')^+$
$$\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$$

- φ respects the scalar multiplication
$$\varphi(c v_1) = c\, \varphi(v_1)$$

(Together: $\varphi(c_1 v_1 + c_2 v_2) = c_1 \varphi(v_1) + c_2 \varphi(v_2)$)

A linear map $V \xrightarrow{\varphi} V'$ is called an (F-vector space) isomorphism if it is also bijective

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One more obvious concept: A subspace $U \subset V$ an F-vector space is a subset closed under $+$ and scalar multiplication

i.e. $u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U$ and $\left.\begin{matrix} u \in V \\ c \in F \end{matrix}\right\} \Rightarrow cu \in U$

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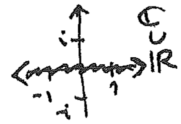
EXAMPLES



(1) $\mathbb{R} \overset{\text{subspace}}{\subset} \mathbb{C} = \{a+bi : a,b\in\mathbb{R}\}$ ($i^2=-1$) $\overset{\text{subspace}}{\subset} \mathbb{H} = \{a+bi+cj+dk : a,b,c,d\in\mathbb{R}\}$ quaternions

$\mathbb{C} \xrightarrow{\varphi} \mathbb{R}^2 = \left\{\begin{bmatrix}a\\b\end{bmatrix} : a,b\in\mathbb{R}\right\}$, $a+bi \mapsto \begin{bmatrix}a\\b\end{bmatrix}$

φ is an isomorphism of $\mathbb{R}$-vector spaces $\mathbb{C}\cong\mathbb{R}^2$

$\mathbb{H} \xrightarrow{\psi} \mathbb{R}^4 = \left\{\begin{bmatrix}a\\b\\c\\d\end{bmatrix} : a,b,c,d\in\mathbb{R}\right\}$

ψ is an isomorphism $\mathbb{H}\cong\mathbb{R}^4$

(2) $\left\{\begin{matrix}\text{constant functions} : a\in\mathbb{R}\\ f(x)=a\\ f:\mathbb{R}\to\mathbb{R}\end{matrix}\right\} \overset{\text{subspace}}{\subset} \left\{\begin{matrix}\text{(affine-) linear functions}\\ f(x)=a_1x+a_0\\ f:\mathbb{R}\to\mathbb{R}\end{matrix}\right\} \overset{\text{subspace}}{\subset} \left\{\begin{matrix}\text{quadratic}\\ f(x)=a_2x^2+a_1x+a_0\\ f:\mathbb{R}\to\mathbb{R}\end{matrix}\right\} \overset{\text{subspace}}{\subset} \ldots \overset{\text{subspace}}{\subset} \left\{\begin{matrix}\text{degree } n \text{ polynomial functions}\\ f(x)=a_nx^n+\ldots+a_2x^2+a_1x+a_0\\ f:\mathbb{R}\to\mathbb{R}\end{matrix}\right\}$

$f(x) \mapsto a$, isomorphism φ to $\mathbb{R}$

isomorphism φ to $\mathbb{R}^2$, $f(x)\mapsto\begin{bmatrix}a_0\\a_1\end{bmatrix}$

$\mathbb{R}^{n+1}$, $f(x)\mapsto\begin{bmatrix}a_0\\a_1\\\vdots\\a_n\end{bmatrix}$

$\overset{\text{subspace}}{\subset} V = \left\{\begin{matrix}\text{all functions}\\ f:\mathbb{R}\to\mathbb{R}\end{matrix}\right\}$ with pointwise $+$ and scaling

i.e. $f+g := f(x)+g(x)$

$cf := cf(x)$

(3) $\left\{\begin{matrix}\text{solutions}\\ y=f(x) \text{ to}\\ \frac{d^2y}{dx^2}=-y\end{matrix}\right\} \overset{\text{subspace}}{\subset} \left\{\begin{matrix}\text{twice-differentiable}\\ f:\mathbb{R}\to\mathbb{R}\end{matrix}\right\} \overset{\text{subspace}}{\subset} V$

(4) Nullspaces of matrices $A\in\mathbb{F}^{m\times n}$

$= \{\text{solutions } X\in F^n \text{ to } AX=0\}$

since can add them, scale them:

$A(X_1+X_2) = AX_1+AX_2 = 0+0=0$ if $AX_1=AX_2=0$

$A(cX)=c\,AX=c\cdot 0=0$

(5) In F^2, the only subspaces are $\{0\}$, F^2 itself and lines through $\underline{0}$

(see Artin's PROP 3.1.4 & 3.3.3 for careful proof, but it will be easier a bit later)

$\ell = \left\{c\begin{bmatrix}a\\b\end{bmatrix} : c\in F\right\}$ for some $\begin{bmatrix}a\\b\end{bmatrix}\neq\begin{bmatrix}0\\0\end{bmatrix}$

$= \left\{\text{solutions } \begin{bmatrix}x\\y\end{bmatrix} \text{ to } [a\ b]\begin{bmatrix}x\\y\end{bmatrix}=[0]\right\}$, $\quad [a\ b]\begin{bmatrix}x\\y\end{bmatrix} = ax+by$

e.g.