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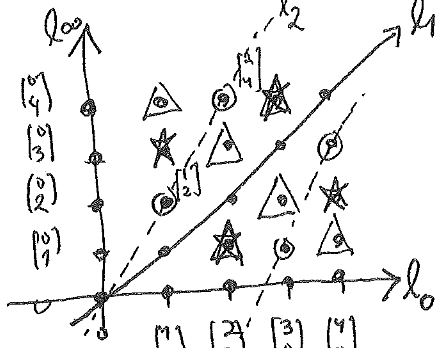
⑥ How many lines through $\underline{0}$ in $\mathbb{F}_5^2$? In $\mathbb{F}_p^2$?

(EXAMPLE 3.3.4)



⊙ $= \ell_2$
★ $= \ell_3$
△ $= \ell_4$

$\left\{\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{F}_5^2 : y = mx\right\} = \ell_m = \mathbb{F}_5 \cdot \begin{bmatrix}1\\m\end{bmatrix} = \left\{ c\begin{bmatrix}1\\m\end{bmatrix} : c \in \mathbb{F}_5 \right\}$

"slope m"

for $m = 0,1,2,3,4$

$\ell_\infty = \mathbb{F}_5 \begin{bmatrix}0\\1\end{bmatrix}$

$5+1=6$ lines

There are p^2-1 vectors $v \neq \begin{bmatrix}0\\0\end{bmatrix}$, and on each line ℓ there are $p-1$ of these vectors $v \neq \begin{bmatrix}0\\0\end{bmatrix}$ that each span the same line ℓ.

Thus there should be

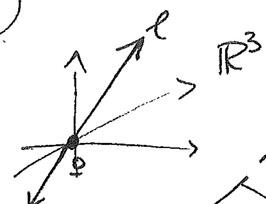
$$\frac{p^2-1}{p-1} \text{ lines} = p+1 \overset{p=5}{\rightsquigarrow} 5+1 = 6$$

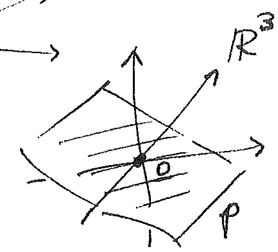
agreeing with picture at left.

§3.4 Bases, dimension, linear independence, spanning

Let's make sense of subspaces in $\mathbb{R}^3$ (or F^3)

being
- 0-dimensional ($\{\underline{0}\}$)
- 1-dimensional (lines ℓ)
- 2-dimensional (planes p)
- 3-dimensional ($\mathbb{R}^3$ itself)





DEF'N: Given $S = \{v_1, \ldots, v_n\} \subset V$, a linear combination of S is a vector $c_1v_1 + \cdots + c_nv_n \in V$ with $c_i \in F$

The span of S, $\text{span}_F(S) := \{$ all linear combinations $c_1v_1 + \cdots + c_nv_n : c_i \in F \}$,

a subspace of V

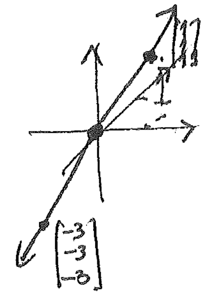
Why?

EXAMPLES: ① Given $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \in F^{m \times n}$ with columns $v_1, \ldots, v_n \in F^m = V$,

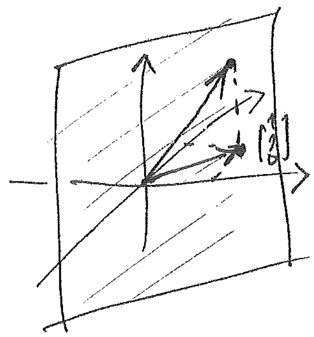
its column space is $\text{span}_F(v_1, \ldots, v_n)$

$= \{x_1v_1 + \cdots + x_nv_n : x_i \in F\} = \{AX : X = \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} \in F^n\}$

② Inside $\mathbb{R}^3$, $\mathrm{span}_{\mathbb{R}}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \left\{\begin{bmatrix}c\\c\\c\end{bmatrix} : c\in\mathbb{R}\right\}$ is a line



$\mathrm{Span}_{\mathbb{R}}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right)$ is a plane



DEF'N: Given $S = \{v_1, \ldots, v_n\} \subseteq V$, say S is (linearly) dependent

if $\exists\, c_i \in F$ not all zero with $c_1 v_1 + \cdots + c_n v_n = \underline{0}$

i.e. $\exists\, \underline{c} = \begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix} \neq \underline{0}$ with $c_1 v_1 + \cdots + c_n v_n = \underline{0}$

i.e. a nonzero solution $\underline{c}$ to $\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix} = \begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix}$

Otherwise one says $S = \{v_1, \ldots, v_n\}$ are (linearly) independent,

i.e. if $c_1 v_1 + \cdots + c_n v_n = \underline{0}$ implies $c_1 = \cdots = c_n = 0$

i.e. $\underline{c} = \begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix} = \underline{0}$

$\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix}c_1\\ \vdots \\ c_n\end{bmatrix}$

EXAMPLES: ① When is $S = \{v_1\}$ dependent?

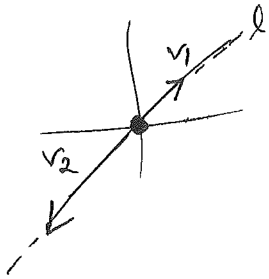
(If and) Only if $v_1 = \underline{0}$: if $v_1 = \underline{0}$ then any $c_1 \in F^\times$ has $c_1 \cdot v_1 = c \cdot \underline{0} = \underline{0}$

if $v_1 \neq \underline{0}$, ~~get a contradiction~~ and $c_1 v_1 = 0$ then either $c_1 = 0$ (done)

or $c_1 \neq 0$, so $c_1^{-1} \cdot c_1 v_1 = c_1 \underline{0} = \underline{0}$ contradiction

$v_1 = 1 \cdot v_1$

② Two vectors $S = \{v_1, v_2\}$ are dependent $\Leftrightarrow$ they lie on common line through $\{\underline{0}\}$



To show $\Leftarrow$, think about two cases, where both are nonzero or at least one is $\underline{0}$

To show $\Rightarrow$, given $c_1 v_1 + c_2 v_2$ wit $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

then assuming $c_1 \neq 0$, one has $v_1 = \frac{-c_2}{c_1} v_2$;

$v_2 = \frac{-c_1}{c_2} v_1$ if $c_2 \neq 0$

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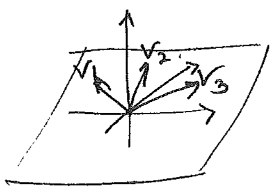
③ Three vectors $S = \{v_1, v_2, v_3\}$ in $\mathbb{R}^3$ are dependent $\Leftrightarrow$ coplanar



④ $S = \emptyset$ is considered independent.

Note that testing dependence of $S = \{v_1, \dots, v_n\}$ is another role for row-reduction

e.g. are $S = \left\{ \begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{array} \right\}$ dependent in $\mathbb{R}^4$? in $\mathbb{F}_2^4$?

That is, is there a nonzero solution $\underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to $c_1 v_1 + c_2 v_2 + c_3 v_3 = \underline{0}$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{\text{over } \mathbb{R}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{c} = \underline{0}$$

independent

over $\mathbb{F}_2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \underline{c} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix}, \; x_3 \in \mathbb{F}_2$$

$$= x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

e.g. $\underline{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq \underline{0}$

dependent

DEF'N (3.4.13): A <u>basis</u> for a vector space V is a subset $B = \{v_1, \ldots, v_n\} \subset V$

that is • <u>linearly independent</u>

<u>and</u> • <u>spans</u> V, i.e. $\mathrm{span}_F B = V$

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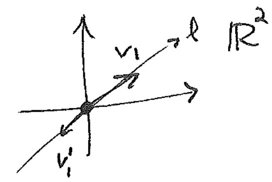
<u>EXAMPLE</u>: ① The <u>standard basis</u> for F^n is $B = \{e_1, e_2, \ldots, e_n\}$

$$e_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix},\quad e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\quad \ldots,\quad e_n = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$$

② A line ℓ through $\underline{0}$ in $\mathbb{R}^n$ has <u>any</u> vector $v_1 \in \ell - \{\underline{0}\}$ giving a basis $B = \{v_1\}$ for ℓ



③ The subspace $\{\underline{0}\}$ of V has $B = \emptyset$ as a basis‽ Linearly independent, <u>yes</u>.

Does it <u>span</u> V?

Well, $\underline{0}$ is the <u>empty</u> <u>sum</u> of vectors(‽)

So, <u>yes</u>.

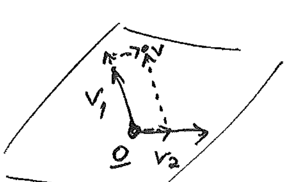
PROPOSITION (PROP 3.4.14): A subset $B = \{v_1, \ldots, v_n\}$ is a <u>basis</u> for V

$\iff$ every $v \in V$ <u>has</u> a <u>unique</u> expression $v = c_1 v_1 + \ldots + c_n v_n$

existence uniqueness

<u>proof</u>: In fact, every $v \in V$ has <u>some</u> (possibly not unique) expression $v = \sum_{i=1}^n c_i v_i \iff B = \{v_1, \ldots, v_n\}$ spans V,

by definition of spanning.



But given B spans V, why should <u>uniqueness</u> of expansions $v = \sum_{i=1}^n c_i v_i$ be equivalent to linear independence?

Certainly <u>dependence</u> $\underline{0} = c_1 v_1 + \ldots + c_n v_n$ with $\underline{c} \neq \underline{0}$ gives two different expressions for $\underline{0}$, $\sum_{i=1}^n 0 \cdot v_i \neq \sum_{i=1}^n c_i v_i$

But two <u>different</u> expressions $\sum_{i=1}^n c_i v_i = v = \sum_{i=1}^n c_i' v_i$ also gives a <u>dependence</u> $\underline{0} = \sum_{i=1}^n (c_i' - c_i) v_i$ not <u>all</u> zero! ∎