(127)

DEF'N (3.4.13) : A *basis* for a vector space V is a subset $B=\{v_1,\ldots,v_n\} \subset V$

that is • *linearly independent*

and • *spans* V , i.e. $\mathrm{span}_F B = V$

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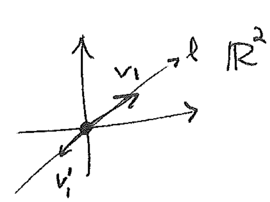
EXAMPLE: ① The *standard basis* for F^n is $B=\{e_1, e_2, \ldots, e_n\}$

$$e_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix},\quad e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\quad e_n = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$$

② A line ℓ through $\underline{0}$ in $\mathbb{R}^n$ has *any* vector $v_1 \in \ell - \{\underline{0}\}$ giving a basis $B=\{v_1\}$ for ℓ



③ The subspace $\{\underline{0}\}$ of V has $B=\emptyset$ as a basis! Linearly independent, *yes*.

Does it *span* V?

Well, $\underline{0}$ is the *empty sum* of vectors(!)

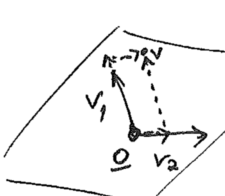
So, *yes*.

PROPOSITION (PROP 3.4.14) : A subset $B=\{v_1,\ldots,v_n\}$ is a *basis* for V

$\iff$ every $v \in V$ *has* a *unique* expression $v=c_1v_1+\ldots+c_nv_n$

existence uniqueness

proof: In fact, every $v\in V$ has *some* (possibly not unique) expression $v=\sum_{i=1}^n c_iv_i \iff B=\{v_1,\ldots,v_n\}$ spans V,



by definition of spanning.

But given B spans V, why should *uniqueness* of expansions $v=\sum_{i=1}^n c_iv_i$ be equivalent to linear independence?

Certainly *dependence* $\underline{0}=c_1v_1+\ldots+c_nv_n$ with $\underline{c}\neq\underline{0}$ gives two *different* expressions for $\underline{0}$, $\sum_{i=1}^n 0\cdot v_i \neq \sum_{i=1}^n c_iv_i$

But two *different* expressions $\sum_{i=1}^n c_iv_i = v = \sum_{i=1}^n c_i'v_i$ also gives a *dependence*

$$\underline{0} = \sum_{i=1}^n \underbrace{(c_i'-c_i)}_{\text{not all zero!}} v_i \quad \blacksquare$$

REMARK: In fact, this shows there are lots of ways to characterize $B = \{v_1, \ldots, v_n\}$ being a basis for $V = F^n$

$\Leftrightarrow$ $\forall Y \in F^n$ $\exists$ a unique expression $Y = \sum_{i=1}^{n} c_i v_i$

$\Leftrightarrow$ $\forall Y \in F^n$, $AX = Y$ has a unique solution $X = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, where $A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \in F^{n \times n}$

$\Leftrightarrow$ A invertible

$\Leftrightarrow$ $\det A = 0$

$\vdots$ (others from Chap. 1)

DEF'N: Say an F-vector space V is finite-dimensional if $\exists$ some finite $S = \{v_1, \ldots, v_n\}$ that spans V i.e. $\mathrm{span}_F(S) = V$.

We'll generally deal only with such V here, starting with a basic result:

THEOREM: (PROP 3.4.16) Given a finite-dimensional F-vector space V with a finite spanning set $S \supseteq L$ an independent set, one can always find a basis B for V with $L \subseteq B \subseteq S$

(In particular, taking $L = \emptyset$, one can always cut S down to a basis for V by removing some elements. So V has some basis!)

proof: Let B be a subset of S containing L that is independent, and maximal under inclusion with respect to this property, meaning $B \cup \{v\}$ is dependent $\forall v \in S - B$.

Certainly B is independent, but we CLAIM it also has $\mathrm{span}_F(B) = \mathrm{span}_F(S) = V$, so it is a basis for V.

(129) To see why, let's go back and prove...

PROPOSITION: (PROP 3.4.15(b)) If $\{v_1, \ldots, v_n\}$ are independent, then

$\{v_1, \ldots, v_n, v_{n+1}\}$ is dependent $\iff v_{n+1} \in \operatorname{span}\{v_1, \ldots, v_n\}$.

proof: Assume $\{v_1, \ldots, v_n\}$ independent.

Then $\{v_1, \ldots, v_n, v_{n+1}\}$ dependent

$\iff \exists\, c_1, \ldots, c_n, c_{n+1} \in F$, not all 0, with

$c_1 v_1 + \cdots + c_n v_n + c_{n+1} v_{n+1} = \underline{0}$

$\iff \exists$ such c_i's $\underline{\text{with } c_{n+1} \neq 0}$

and $c_1 v_1 + \cdots + c_n v_n + c_{n+1} v_{n+1} = \underline{0}$

i.e. $v_{n+1} = \sum_{i=1}^{n} \left(\frac{-c_i}{c_{n+1}}\right) v_i$ let $c_i' := \frac{-c_i}{c_{n+1}} \in F$

i.e. $v_{n+1} = \sum_{i=1}^{n} c_i' v_i$

$\iff v_{n+1} \in \operatorname{span}_F\{v_1, \ldots, v_n\}$ ∎

Now if $\underset{\text{independent}}{L} \subseteq \underset{\text{indep.}}{B} \subseteq \underset{\text{spanning } V}{S}$ and $B \cup \{v\}$ is dependent $\forall v \in S - B$,

then by the above PROP, one has $\overset{v_{n+1}}{v} \in \operatorname{span}_F(\overset{\{v_1, \ldots, v_n\}}{B})$ $\forall v \in S - B$,

and already $v \in \operatorname{span}_F(B)$ $\forall v \in B$,

so $S \subseteq \operatorname{span}_F(B)$,

$\Rightarrow \underbrace{\operatorname{span}_F(S)}_{= V} \subseteq \operatorname{span}_F(B)$, i.e. B spans V, and hence is a basis for V. ∎

Now that we know bases B exist in finite-dimensional V,

let's see why they have same size...

THEOREM-DEF'N: If an F-vector space V has a finite spanning set S, then every linearly independent subset $L \subset V$ has $|L| \leq |S|$.

In particular, any two bases B, B' for V have same cardinality $|B| = |B'|$, called the dimension $\dim(V) = \dim_F(V)$.

proof: Let $S = \{v_1, \ldots, v_m\}$, so $m = |S|$.

Given any independent set $L \subset V$, if $|L| > m$ (e.g. $|L| = \infty$, or $|L| = n > m$)

then one can pick a finite subset $\{w_1, \ldots, w_n\} \subset L$ with $n > m$, and $\{w_1, \ldots, w_n\}$ is also independent (Why?).

Then we'll get a contradiction to $n > m$ as follows.

Express each $w_j = \sum_{i=1}^{m} a_{ij} v_i$ for $j = 1, \ldots, n$ with $A = (a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} \in F^{m \times n}$

$$= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

and since $n > m$, A has some $\underline{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \neq \underline{0}$ in its nullspace

i.e. $A\underline{c} = \underline{0}$ has a nontrivial solution.

But then we claim $c_1 w_1 + \ldots + c_n w_n = 0$ gives a dependence on $\{w_1, \ldots, w_n\}$,

since $$\sum_{j=1}^{n} c_j w_j = \sum_{j=1}^{n} c_j \left(\sum_{i=1}^{m} a_{ij} v_i \right) = \sum_{i=1}^{m} \underbrace{\left(\sum_{j=1}^{n} a_{ij} c_j \right)}_{(A\underline{c})_i = 0} v_i = \sum_{i=1}^{m} 0 \cdot v_i = 0 \quad \blacksquare$$