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A seemingly obvious property of $\dim(V)$ is a bit subtle (and comes up a lot).

PROPOSITION (3.4.23): If V is finite dimensional, and $U \subseteq V$ is a subspace, then U is also finite-dimensional, $\dim U \leq \dim V$, and equality holds $\Leftrightarrow U = V$.

Proof: Build a basis for U one step at a time by starting with the $\emptyset$ indep. set, and adding vectors until it spans U.

That is, let $\{u_1, \ldots, u_k\} \subset U$ be independent, and if $U \supsetneq \mathrm{span}_F\{u_1, \ldots, u_k\}$ then pick $u_{k+1} \in U \setminus \mathrm{span}_F\{u_1, \ldots, u_k\}$, so $\{u_1, \ldots, u_k, u_{k+1}\}$ is still independent, in U, and in V.

Hence this stops before k reaches $n+1$, so it must stop with a basis $\{u_1, \ldots, u_k\}$ of U having $k \leq n$, i.e. $\dim U \leq \dim V$ (and it's finite).

If $k = n$, then this basis $\{u_1, \ldots, u_n\}$ for U must also span V, else one has some $v \in V - \mathrm{span}\{u_1, \ldots, u_n\}$ that one could add to get $\{u_1, \ldots, u_n, v\}$ independent of size $n+1$ in V; contradiction.

Hence $V = \mathrm{span}\{u_1, \ldots, u_k\} = U$ ∎

§3.5 Computing with bases

Picking an ordered basis $B = (v_1, \ldots, v_n)$ for an F-vector space V let's one identify V with F^n.

PROPOSITION (3.5.4): Given an ordered set $(v_1, \ldots, v_n) \subset V$,

the map $F^n \xrightarrow{\psi} V$.

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \longmapsto \psi(\underline{x}) = x_1 v_1 + \cdots + x_n v_n$$

is always F-linear, and

is injective $\iff \{v_1, \ldots, v_n\}$ independent

surjective $\iff \{v_1, \ldots, v_n\}$ spanning V

and thus bijective (so an F-vector space isomorphism) $\iff \{v_1, \ldots, v_n\}$ is a basis for V.

In this case, call $\psi^{-1}(v) = \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, the coordinates of v with respect to $(v_1, \ldots, v_n) = B$

proof: Linearity is pretty clear – check $\psi(a\underline{x} + b\underline{x}') = a\psi(\underline{x}) + b\psi(\underline{x}')$

Since linear maps are group homomorphisms $(F^n)^+ \xrightarrow{\psi} V^+$,

ψ is injective $\iff \ker \psi = \{\underline{0}\} \iff x_1 v_1 + \cdots + x_n v_n = \underline{0}$ implies $\underline{x} = \underline{0}$

$\iff \{v_1, \ldots, v_n\}$ independent.

The rest proves itself! ∎

EXAMPLE: Recall $V = \{$twice differentiable $y(t)$ solving $y'' + y = 0\}$

was an $\mathbb{R}$-vector space under pointwise $+$ and $\mathbb{R}$-scaling.

In differential equations, one considers the basic solutions (or fundamental)

$y_1 = \cos(t)$, $y_2 = \sin(t)$

with $\begin{bmatrix} y_1(0) \\ y_1'(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} y_2(0) \\ y_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} \cos(0) \\ -\sin(0) \end{bmatrix}$ $\begin{bmatrix} \sin(0) \\ \cos(0) \end{bmatrix}$

, and shows every solution y can be written uniquely as

$y = c_1 y_1 + c_2 y_2$ where $c_1 = y(0)$, $c_2 = y'(0)$

(The idea is that then $f(t) = y(t) - (c_1 y_1(t) + c_2 y_2(t))$ has $f''+f=0$
$f(0)=0$
$f'(0)=0$
forcing $f(t) \equiv 0 \ \forall t$)

In other words $B = (y_1 = \cos t, y_2 = \sin(t))$ is an ordered basis for V,

and hence one has a $\mathbb{R}$-vector space isomorphism

$$\mathbb{R}^2 \xrightarrow{\psi} V \qquad \text{, showing } V \cong \mathbb{R}^2.$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto x_1 \cos(t) + x_2 \sin(t)$$

Suppose someone suggests a different subset $(v_1', \ldots, v_n') = B' \subset V$ might be a basis. How to check if it is, and if so how to relate the coordinates $\underline{x}'$ for v with respect to $B' = (v_1', \ldots, v_n')$ and coordinates $\underline{x}$ for v — " — $B = (v_1, \ldots, v_n)$?

<u>PROPOSITION</u>: (3.5.9) Given ordered basis $B = (v_1, \ldots, v_n) \subset V$
and n vectors $B' = (v_1', \ldots, v_n') \subset V$,
define the change-of-basis matrix $P = (p_{ij}) \in F^{n \times n}$ by uniquely expressing $v_j' = \sum_{i=1}^{n} p_{ij} v_i$
for $j = 1, \ldots, n$

Then (i) B' is <u>another</u> basis $\Leftrightarrow$ P is invertible

and (ii) If so, then a vector $v \in V$ with coordinates $\underline{x}$ in B
and $\underline{x}'$ in B

will have $P\underline{x}' = \underline{x}$
and $\underline{x}' = P^{-1}\underline{x}$

proof: For (i), if P is invertible and $P^{-1} = (p^{-1}_{ij})$ then $\{v_1', \dots, v_n'\}$ span V

since $$\sum_{i=1}^{n} p^{-1}_{ij} v_i' = \sum_{i=1}^{n} p^{-1}_{ij} \left(\sum_{k=1}^{n} p_{ki} v_k \right) = \sum_{k=1}^{n} \underbrace{\left(\sum_{i=1}^{n} p_{ki} p^{-1}_{ij} \right)}_{(P \cdot P^{-1})_{kj} = I_{kj}} v_k = v_j$$

and this is reversible: if $\{v_1', \dots, v_n'\}$ span V

then $\exists\, (q_{ij})$ with $$v_j = \sum_{i=1}^{n} q_{ij} v_i' = \sum_{i=1}^{n} q_{ij} \left(\sum_{k=1}^{n} p_{ki} v_k \right) = \sum_{k=1}^{n} \underbrace{\left(\sum_{i=1}^{n} p_{ki} q_{ij} \right)}_{\substack{\text{must be} \\ \left\{ \begin{smallmatrix} 1 & \text{if } k=j \\ 0 & \text{else} \end{smallmatrix} \right\}}} v_k$$

since $v_1, \dots, v_n$ are independent

Thus $Q = (q_{ij}) = P^{-1}$.

So $\{v_1', \dots, v_n'\}$ span $V \iff P$ invertible

hence they're a basis $\iff P$ invertible (since $\dim V = n$)

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For (ii), if $v \in V$ has coordinates $\underline{x}$ for B

then $$v = x_1 v_1 + \dots + x_n v_n = \sum_{i=1}^{n} x_i v_i$$

and hence $\underline{x}' = P^{-1} \underline{x}$ has

$$x_1' v_1' + \dots + x_n' v_n' = \sum_{i=1}^{n} x_i' v_i' = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p^{-1}_{ij} x_j \right) \left(\sum_{k=1}^{n} p_{ki} v_k \right)$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \underbrace{\underbrace{\left(\sum_{i=1}^{n} p_{ki} p^{-1}_{ij} \right)}_{(P \cdot P^{-1})_{kj}}}_{I_{kj}} x_j \right) v_k$$

$$= \sum_{k=1}^{n} x_k v_k = v \quad \blacksquare$$