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LEMMA: Every word w in $S \sqcup S^{-1}$ has a unique reduction w_{red} which is reduced.
(PROP 7.9.2)

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proof: Induct on ℓ in $w = (x_1, x_2, \ldots, x_\ell)$ for both the existence & uniqueness.
(Existence is clear by induction.)
In the base case where w is already reduced, $w_{red} = w$ is unique!

In the inductive step, assume w has two different reductions w_r, w_s', both reduced:

$$\begin{tikzcd}[column sep=small]
 & w_1 \arrow[r] & w_2 \arrow[r] & \cdots \arrow[r] & w_r \\
w \arrow[ur] \arrow[dr] & & & & \\
 & w_1' \arrow[r] & w_2' \arrow[r] & \cdots \arrow[r] & w_s' \arrow[uu, dash, "\neq"]
\end{tikzcd}$$

CASE 1: $w_1 = w_1'$ (which could happen either as $w = Axx^{-1}B \rightarrow w_1 = AB \rightarrow \ldots$, $\rightarrow w_1' = AB \rightarrow \ldots$

or as $w = Axx^{-1}xB \rightarrow w_1 = AxB \rightarrow \ldots$, $\rightarrow AxB \rightarrow \ldots = w_1'$)

But then w_1 is shorter than w and has two different reduced reductions; contradiction to inductive hypothesis.

CASE 2: $w_1 \neq w_1'$

So one must have this picture:

$$\begin{tikzcd}[column sep=small]
 & w_1 = ABy^{-1}yC \arrow[r] \arrow[dr, dotted] & w_2 \arrow[r] & \cdots \arrow[r] & w_r \\
w = Axx^{-1}By^{-1}yC \arrow[ur] \arrow[dr] & & ABC \arrow[r, dotted] & \cdots \arrow[r] & w_t'' \\
 & w_1' = Axx^{-1}BC \arrow[ur, dotted] \arrow[r] & w_2' \arrow[r] & \cdots \arrow[r] & w_s'
\end{tikzcd}$$

all 3 reduced, and possibly $w_t'' = w_r$ or $w_t'' = w_s'$

But since either $w_r \neq w_t''$, or $w_s' \neq w_t''$, either w_1 or w_1' has different reductions that are both reduced; contradiction again ■

DEF'N: The free group $F(S)$ on a set S is the collection of all equivalence classes of words in $S \sqcup S^{-1}$ for the equiv. relation $w \sim w'$ if $w_{red} = w'_{red}$

with multiplication
$$F(S) \times F(S) \longrightarrow F(S)$$
$$([u], [v]) \longmapsto [uv]$$
$$u = u_1 \cdots u_\ell, \quad v = v_1 \cdots v_m, \quad uv = u_1 \cdots u_\ell v_1 \cdots v_m$$
called concatenation

PROPOSITION: This multiplication on $F(S)$ is well-defined,
(PROP 7.9.3, 7.9.4)
and makes it a group.

proof: To see it's well-defined, assume $u \sim u'$ so $u_{red} = u'_{red}$
$v \sim v'$ $\quad v_{red} = v'_{red}$

and we want to show $uv \sim u'v'$ i.e. $(uv)_{red} = (u'v')_{red}$.
But this is true since one can reduce both $uv, u'v' \to \to \cdots \to u_{red}v_{red}$,
which might not be reduced, but then $(u_{red}v_{red})_{red}$ must equal
both $(uv)_{red}$ and $(u'v')_{red}$ by the uniqueness LEMMA.

e.g. $(u = aaa^{-1}, \underset{v_{red}=v'_{red}}{v = a^{-1}bb}) \longmapsto uv = aaa^{-1}a^{-1}bb \to \cdots \to bb$
$(\underset{u_{red}=u'_{red}}{u' = a}, \; v' = a^{-1}bbb^{-1}b) \longmapsto u'v' = aa^{-1}bbb^{-1}b \to \cdots \to bb$

$u_{red}v_{red} = aa^{-1}bb \to \cdots \to bb = (u_{red}v_{red})_{red}$

Once it's well-defined,
- associativity of $[(uv)w] = [u(vw)]$ is clear ($= [uvw]$)
- identity in $F(S)$ is $1 = [\,]$ (empty word)
- inverse of $w = x_1x_2\cdots x_\ell$ is $w^{-1} = x_\ell^{-1}x_{\ell-1}^{-1}\cdots x_2^{-1}x_1^{-1}$ ∎

NOTE: We'll drop the brackets $[w]$ around w when working in $F(S)$
e.g. writing $a^2a^{-3}b = a^{-1}b$ in $F(\{a,b\})$

REMARK: Although $F(S)$ might seem easy to understand, it can be tricky (once $|S| \geq 2$)
e.g. EXERCISE 7.9.1 asks you to show that inside $F(\{x,y\})$,
the subgroup $\langle x^2, y^2, xy \rangle \cong F(\{u,v,w\})$ (!)

An important tool in using $F(S)$ is this universal property:

PROPOSITION: Given any group G and a map of the set $S \xrightarrow{f} G$, $s_i \longmapsto f(s_i)$
(PROP 7.10.12)
one can extend f to a group homomorphism $F(S) \xrightarrow{\varphi} G$ such that $s_i \longmapsto \varphi(s_i) = f(s_i)$
and this extension φ is unique.

proof: Once one knows $\varphi(s_i) = f(s_i) \in G$, then φ being a homomorphism forces $\varphi(s_1^{\pm 1} s_2^{\pm 1} \cdots s_\ell^{\pm 1}) = \varphi(s_1^{\pm 1}) \cdots \varphi(s_\ell^{\pm 1}) = f(s_1)^{\pm 1} \cdots f(s_\ell)^{\pm 1}$.

On the other hand, this definition for φ is well-defined, since if $[\omega] = [\omega']$ in $F(S)$ then $\omega_{red} = \omega'_{red}$, and one can check that $\varphi(\omega) = \varphi(\omega_{red})$ since each cancellation $Axx^{-1}B \to AB$

has
$$\varphi(Axx^{-1}B) = \varphi(A)\varphi(x)\varphi(x)^{-1}\varphi(B) = \varphi(A)\varphi(B) = \varphi(AB) \quad \blacksquare$$

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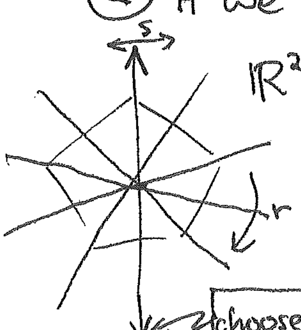
EXAMPLES:

① Exercise 7.9.1 is asking one to show that the unique homomorphism
$$F(\{u,v,\omega\}) \xrightarrow{\varphi} F(\{x,y\})$$
defined by
$$u \xmapsto{f} x^2$$
$$v \xmapsto{f} y^2$$
$$\omega \xmapsto{f} xy$$
is an isomorphism onto $\mathrm{im}(\varphi) = \langle x^2, y^2, xy \rangle < F(\{x,y\})$

② If we define $D_n := \{$linear symmetries of regular n-gon$\}$



$$= \left\langle \overset{\text{"s"}}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}, \overset{\text{"r"}}{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \right\rangle < GL_2(\mathbb{R})$$
where $\theta = \frac{2\pi}{n}$

choose this line to be the y-axis in $\mathbb{R}^2$

then there is a unique homomorphism
$$F(\{r,s\}) \xrightarrow{\varphi} D_n$$
defined by
$$s \xmapsto{f} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$r \xmapsto{f} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Q: Who lies in $K := \ker(\varphi)$?
$s^2 = s \cdot s$
$r^n = r \cdot r \cdots r$
$srsr$ (because we had $srs = r^{-1}$ in D_n i.e. $srsr = 1$)

and conjugates of them, like As^2A^{-1}, Ar^nA^{-1}, $AsrsrA^{-1}$ since $K \lhd F(\{r,s\})$.
and products of these and their inverses