proof: Once one knows $\varphi(s_i) = f(s_i) \in G$, then φ being a homomorphism forces $\varphi(s_1^{\pm 1} s_2^{\pm 1} \cdots s_\ell^{\pm 1}) = \varphi(s_1^{\pm 1}) \cdots \varphi(s_\ell^{\pm 1}) = f(s_1)^{\pm 1} \cdots f(s_\ell)^{\pm 1}$.

On the other hand, this definition for φ is well-defined, since if $[\omega] = [\omega']$ in $F(S)$ then $\omega_{red} = \omega'_{red}$, and one can check that $\varphi(\omega) = \varphi(\omega_{red})$ since each cancellation $Axx^{-1}B \to AB$ has

$$\begin{aligned}\varphi(Axx^{-1}B) &= \varphi(A)\varphi(x)\varphi(x)^{-1}\varphi(B)\\ &= \varphi(A)\varphi(B)\\ &= \varphi(AB) \quad \blacksquare\end{aligned}$$

11/9/2018 >

EXAMPLES:

① Exercise 7.9.1 is asking one to show that the unique homomorphism

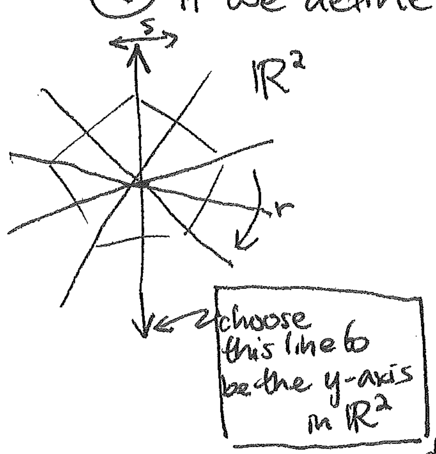
$$F(\{u,v,w\}) \xrightarrow{\varphi} F(\{x,y\})$$

defined by

$$\begin{aligned} u &\xmapsto{f} x^2 \\ v &\xmapsto{f} y^2 \\ w &\xmapsto{f} xy \end{aligned}$$

is an isomorphism onto $\mathrm{im}(\varphi) = \langle x^2, y^2, xy \rangle < F(\{x,y\})$

② If we define $D_n := \{$linear symmetries of regular n-gon$\}$



$$= \left\langle \overset{\text{"s"}}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}, \overset{\text{"r"}}{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \right\rangle < GL_2(\mathbb{R})$$

where $\theta = \frac{2\pi}{n}$

then there is a unique homomorphism

$$F(\{r,s\}) \xrightarrow{\varphi} D_n$$

defined by

$$\begin{aligned} s &\xmapsto{f} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ r &\xmapsto{f} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \end{aligned}$$

Q: Who lies in $K := \ker(\varphi)$?

$s^2 = s \cdot s$

$r^n = r \cdot r \cdots r$

$srsr$ (because we had $srs = r^{-1}$ in D_n i.e. $srsr = 1$)

and conjugates of them, like As^2A^{-1}, Ar^nA^{-1}, $AsrsrA^{-1}$ since $K \triangleleft F(\{r,s\})$, and products of these and their inverses

(109) We need a notion of <u>normal subgroup</u> generated by a subset of a group.

PROP: Given $A = \{a_i\} \subset G$ a group,

- $\underset{\substack{\text{subgroup}\\ \text{of } G\\ \text{generated}\\ \text{by } A}}{\langle A \rangle} := \{\text{all products } a_1^{\pm 1} a_2^{\pm 1} \cdots a_\ell^{\pm 1} : a_i \in A\} = \displaystyle\bigcap_{\substack{\text{subgroups } H < G:\\ H \supseteq A}} H$

- $\underset{\substack{\text{normal}\\ \text{subgroup}\\ \text{of } G\\ \text{generated}\\ \text{by } A}}{\langle A \rangle_{\text{normal}}} := \{\text{all products } g_1 a_1^{\pm 1} g_1^{-1} \cdot g_2 a_2^{\pm 1} g_2^{-1} \cdot \ldots \cdot g_\ell a_\ell^{\pm 1} g_\ell^{-1} : a_i \in A, g \in G\}$

$$= \bigcap_{\substack{\text{normal}\\ \text{subgroups } K \lhd G:\\ K \supseteq A}} K$$

<u>proof</u>: Let's check the 2nd one; 1st is similar.
The left side is easily checked to be a normal subgroup of G, and it contains A, so it contains the right side.
On the other hand, every $K \lhd G$ with $K \supseteq A$ will contain the left side also, so left side is contained in the right side.
So they're equal. ∎

DEF'N: Given a set $S = \{s_1, \ldots, s_n\}$ and a set of relations $R = \{r_1, \ldots, r_\ell\} \subset F(S)$, the <u>group generated by S subject to relations R</u> is

$$\langle s_1, \ldots, s_n \mid r_1, \ldots, r_\ell \rangle := \underset{\substack{\parallel\\ F(\{s_1, \ldots, s_n\})}}{F(S)} \Big/ \underset{\substack{\parallel\\ \langle \{r_1, \ldots, r_\ell\} \rangle_{\text{normal}}}}{\langle R \rangle_{\text{normal}}}$$

One can perform manipulations in this group working with words w and their cosets wK where $K = \langle R \rangle_{\text{normal}}$, replacing r_i's in the middle of words by empty strings, or vice-versa:

$$\underline{u}\, r_i\, \underline{v} K = \underline{u}\, r_i\, \underline{v} \cdot \underbrace{\underline{v}^{-1} r_i^{-1} \underline{v}}_{\in K} K = \underline{u}\, r_i \cdot r_i^{-1} \underline{v} K = \underline{u}\underline{v} K$$

<u>EXAMPLE</u>: The group $\langle r,s \mid s^2, r^n, srsr\rangle$ has at most $2n$ elements $\{\bar{1}, \bar{r}, \bar{r}^2, \ldots, \overline{r^{n-1}}, \bar{s}, \overline{sr}, \ldots, \overline{sr^{n-1}}\}$ where $\bar{w} := wK$ with $w \in F(\{r,s\})$, $K = \langle\{s^2, r^n, srsr\}\rangle_{\text{normal}}$

because one can reduce exponents on r to $\leq n-1$
on s to ≤ 1

via
$$\underline{u}\, r^{n+k}\, \underline{v} K = \underline{u}\, r^k \cdot r^n \underline{v} \cdot \underbrace{\underline{v}^{-1} r^{-n} \underline{v}}_{\in K} K$$
$$= \underline{u}\, r^k \cdot r^n r^{-n} \underline{v} K$$
$$= \underline{u}\, r^k \underline{v} K$$

and one can move s left of any r's via
$$\underline{u}\, rs\, \underline{v} K = \underline{u}\, rs\, \underline{v} \cdot \underbrace{\underline{v}^{-1} s^{-1} r^{-1} s^{-1} r^{-1} \underline{v}}_{\text{this is in } K \text{ because}} K$$

$$srsr \in K \Downarrow s^{-1}(srsr)s \in K \Downarrow rsrs \in K \Downarrow (rsrs)^{-1} \in K \Downarrow s^{-1} r^{-1} s^{-1} r^{-1} \in K$$

$$= \underline{u}\, \cancel{rs s^{-1} r^{-1}}\, s^{-1} r^{-1} \underline{v} K$$
$$= \underline{u}\, s^{-1} r^{-1} \underline{v} K$$

then make exponents nonnegative...
$$= \underline{u} \cdot s^2 \cdot s^{-1} r^{-1} \underline{v} K$$
$$= \underline{u}\, s r^{-1} \underline{v} K$$
$$= \underline{u}\, s r^n \cdot r^{-1} \underline{v} K$$
$$= \underline{u}\, s r^{n-1} \underline{v} K$$

<u>EXAMPLE</u>: If $\langle S \mid R\rangle$ has a relation $\underline{w}_1^{-1}\underline{w}_2$ in R modeling $\underline{w}_1 = \underline{w}_2$,

then in $\langle S \mid R\rangle = F(S) / \underbrace{\langle R\rangle_{\text{normal}}}_{K}$ one has the ability to mak replacements
$$\underline{u}\, \underline{w}_1 \underline{v} K = \underline{u}\, \underline{w}_2 \underline{v} K$$
$$\parallel \qquad\qquad \parallel$$
$$\underline{u}\, \underline{w}_1 \underline{v} \cdot \underbrace{\underline{v}^{-1} \underline{w}_1^{-1} \underline{w}_2 \underline{v}}_{\in K} K = \underline{u}\, \underline{w}_1 \underline{w}_1^{-1} \cdot \underline{w}_2 \underline{v} K$$