But $G = S_n$ also acts on $S := \{\text{ordered pairs } (i,j) : i,j \in \{1,2,\ldots,n\}\}$
$= \{1,2,\ldots,n\} \times \{1,2,\ldots,n\}$

via $p * (i,j) := (p(i), p(j))$

and then there are two orbits $\{(i,i) : i=1,\ldots,n\} = \mathcal{O}_{(1,1)}$, $\{(i,j) : i \neq j\} = \mathcal{O}_{(1,2)}$, so it is not a transitive action.

e.g. $n=4$

$$S = \left\{ \begin{array}{cccc} (1,1), & (1,2), & (1,3), & (1,4), \\ (2,1), & (2,2), & (2,3), & (2,4), \\ (3,1), & (3,2), & (3,3), & (3,4), \\ (4,1) & (4,2), & (4,3), & (4,4) \end{array} \right\} = \mathcal{O}_{(1,2)} = \mathcal{O}_{(1,3)} = \cdots = \mathcal{O}_{(4,3)}$$

$\mathcal{O}_{(1,1)} = \mathcal{O}_{(2,2)} = \cdots$

e.g. $(14) * (1,1) = (4,4)$

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} * (1,3) = (3,4)$

10/17/2018>

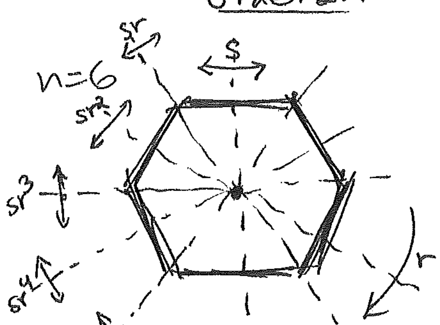
② $D_n :=$ (linear) symmetries of a regular n-sided polygon

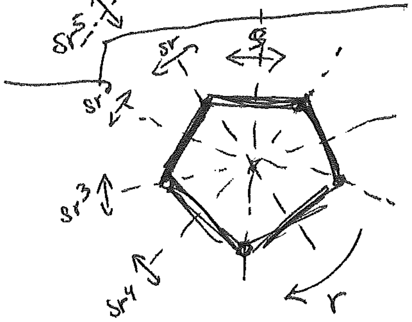
dihedral group of order $2n$

$= \{n \text{ rotations } 1, r, r^2, \ldots, r^{n-1}\} \sqcup \{n \text{ reflections } s, sr, sr^2, \ldots, sr^{n-1}\}$

r = rotation through $\frac{2\pi}{n}$ clockwise

$= \langle r \rangle \sqcup s\langle r \rangle$





$n=5$

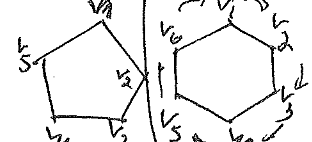
$C_n \cong \langle r \rangle \cong (\mathbb{Z}/n\mathbb{Z})^+$, $\quad \langle r \rangle \triangleleft D_n$

$srs^{-1} = srs = r^{-1} = r^{n-1}$

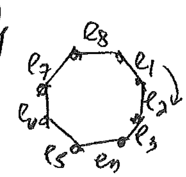
$sr^ks^{-1} = r^{-k}$

$G = D_n$ acts on various sets transitively such as

$S = \{\text{vertices } v_1, \ldots, v_n \text{ of the } n\text{-gon}\}$
$\mathcal{O}_{v_1} = \mathcal{O}_{v_2} = \cdots = \mathcal{O}_{v_n}$



or $S = \{\text{edges } e_1, \ldots, e_n \text{ of the } n\text{-gon}\}$
$\mathcal{O}_{e_1} = \mathcal{O}_{e_2} = \cdots = \mathcal{O}_{e_n}$

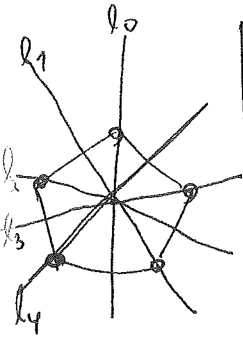


but G also acts nontransitively on $S = \mathbb{R}^2$, since there are infinitely many orbits $\mathcal{O}_s$:

$\mathcal{O}_s \neq \mathcal{O}_{s'}$

(73)

$G = D_n$ acting $S = \{$reflection symmetry lines $\ell_0, \ell_1, \ldots, \ell_{n-1}\}$
(for $s, sr, sr^2, \ldots, sr^{n-1}$)

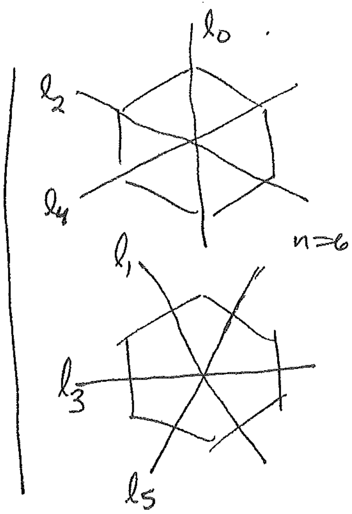


$n=5$

is { **transitive** if n is odd $\mathcal{O}_{\ell_0} = \mathcal{O}_{\ell_1} = \cdots = \mathcal{O}_{\ell_{n-1}}$

{ **nontransitive** if n is even: $\mathcal{O}_{\ell_0} = \mathcal{O}_{\ell_2} = \cdots = \mathcal{O}_{\ell_{n-2}}$

$\neq \mathcal{O}_{\ell_1} = \mathcal{O}_{\ell_3} = \cdots = \mathcal{O}_{\ell_{n-1}}$



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§6.8 / §6.9, The orbit-stabilizer counting formula

DEF'N: Given a group G acting on a set S and $s \in S$,

the **stabilizer subgroup** $G_s := \{g \in G : g(s) = s\}$

PROPOSITION: (PROP 6.7.7, 6.9.2) Two elements $a, b \in G$ have $a(s) = b(s) \iff aG_s = bG_s$.

Consequently, the map $G \xrightarrow{f} \mathcal{O}_s$ (= G-orbit of s)

$a \longmapsto a(s)$

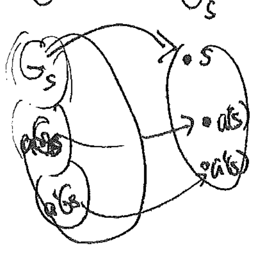
is (surjective and) $|G_s|$-to-1, so that

$$\boxed{|G| = |\mathcal{O}_s| \cdot |G_s|}$$ "Orbit-stabilizer formula"

or $|\mathcal{O}_s| = \dfrac{|G|}{|G_s|} = [G : G_s]$

proof: $a(s) = b(s) \iff \underbrace{a^{-1}a(s)}_{=1(s)=s} = a^{-1}b(s) \iff a^{-1}b \in G_s \iff aG_s = bG_s$

$G \xrightarrow{f} \mathcal{O}_s$



Then the map f has each fiber $\underbrace{f^{-1}(s')}_{= aG_s}$ of size $|aG_s| = |G_s|$ ∎

This might seem a little surprising since it says $|G_s|$ should be the same for every s' having the same G-orbit $\mathcal{O}_{s'} = \mathcal{O}_s$:

$$|G_{s'}| = \frac{|G|}{|\mathcal{O}_{s'}|} = \frac{|G|}{|\mathcal{O}_s|} = |G_s|$$

But something more is true: in this case G_s and $G_{s'}$ will be conjugate subgroups within G, since if $a(s) = s'$ then $aG_s a^{-1} = G_{s'}$

$$\begin{aligned}
(\text{check: } & g \in G_s \iff \\
& g(s) = s \iff \\
& ag(s) = a(s) \iff \\
& aga^{-1} a(s) = a(s) \iff \\
& aga^{-1}(s') = s' \iff \\
& aga^{-1} \in G_{s'})
\end{aligned}$$

EXAMPLES:

① $G = S_n$ acting on $S = \{1, 2, \ldots, n\}$ was transitive, and the one orbit $\mathcal{O}_1 = S$ has stabilizer $G_1 = (S_n)_1 = \{p \in S_n : p(1) = 1\} \cong S_{n-1}$

so $$|S_n| = |\mathcal{O}_1| \cdot |\underbrace{(S_n)_1}_{S_{n-1}}|$$
$$n! = n \cdot (n-1)! \checkmark$$

10/19/2018 > On the other hand, when $G = S_n$ acts on $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$, there were two orbits

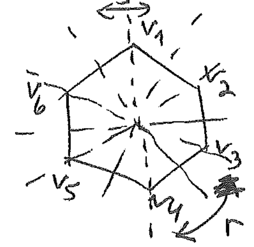
$$\mathcal{O}_{(1,1)} = \{(i,i) : i = 1, 2, \ldots, n\}, \quad G_{(1,1)} \cong S_{n-1}$$
$$\mathcal{O}_{(1,2)} = \{(i,j) : i \neq j\}, \quad G_{(1,2)} \cong S_{n-2}$$

$$|S_n| = |\mathcal{O}_{(1,1)}| \cdot |S_{n-1}|$$
$$n! = n \cdot (n-1)! \text{ again}$$

$$|S_n| = |\mathcal{O}_{(1,2)}| \cdot |S_{n-2}|$$
$$= n(n-1) \cdot (n-2)!$$

(2) $G = D_n$ = dihedral group acting on $S = \{\text{vertices } v_1, \dots, v_n \text{ of } n\text{-gon}\}$

was transitive, and $\mathcal{O}_{v_1}$ has $G_{v_1} = \langle s \rangle = \{1, s\}$
$\mathcal{O}_{v_1} = S = \{v_1, \dots, v_n\}$



$$|D_n| = |\mathcal{O}_{v_1}| \cdot |G_{v_1}|$$
$$2n = n \cdot 2$$

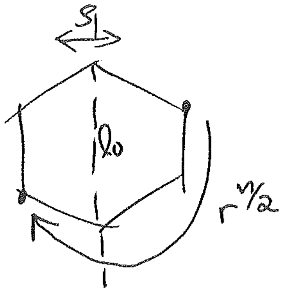
On the other hand, when $G = D_n$ acts on $S = \{\text{symmetry lines } l_0, l_1, \dots, l_{n-1}\}$

it was transitive for n even (because $G_{l_0} = \langle s \rangle$ again)

two orbits $\mathcal{O}_{l_0} = \mathcal{O}_{l_2} = \mathcal{O}_{l_4} = \dots = \mathcal{O}_{l_{n-2}}$ for n odd

$\mathcal{O}_{l_1} = \mathcal{O}_{l_3} = \mathcal{O}_{l_5} = \dots = \mathcal{O}_{l_{n-1}}$

(Q: What is G_{l_0} for n even?)



$$G_{l_0} = \langle s, \underbrace{r^{n/2}}_{\text{rotation by } \pi} \rangle \text{ if } n \text{ even}$$
$$= \{1, s, r^{n/2}, sr^{n/2}\}$$

$$|D_n| = |\mathcal{O}_{l_0}| \cdot |G_{l_0}|$$
$$2n = \begin{cases} n \cdot 2 & \text{if } n \text{ odd,} \\ \frac{n}{2} \cdot 4 & \text{if } n \text{ even.} \end{cases}$$

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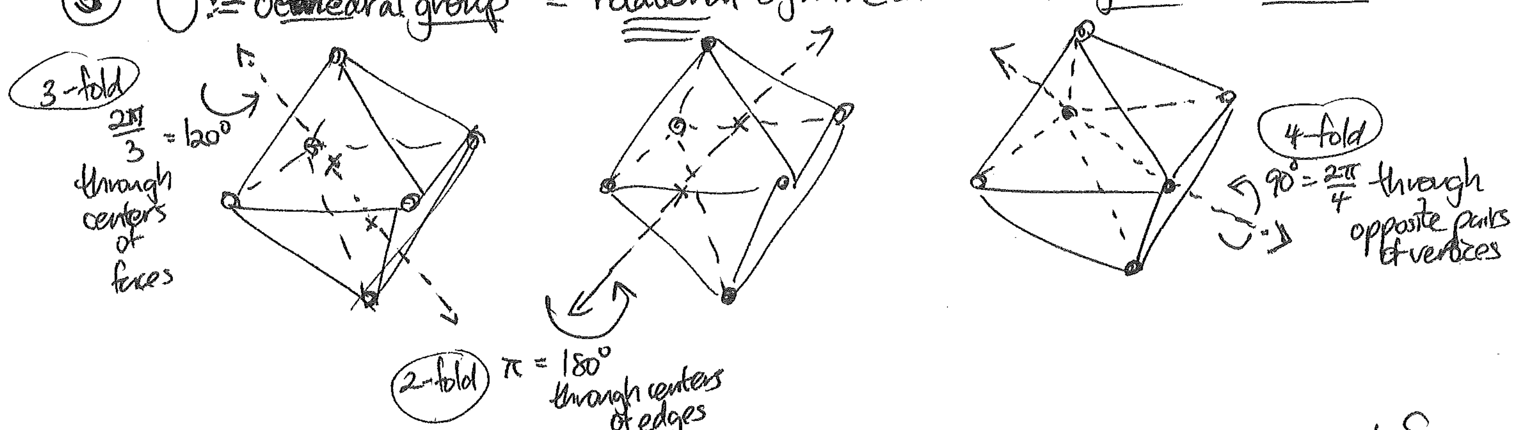
(3) $G = \mathcal{O}$:= octahedral group = (only!) rotational symmetries of a regular octahedron



What is the order of this group $\mathcal{O}$?

$|\mathcal{O}| = |\mathcal{O}_s| \cdot |G_s|$ for various sets S on which $\mathcal{O}$ acts like $S = \{\text{vertices}\}$, $S = \{\text{edges}\}$, $S = \{\text{faces}\}$
$|\mathcal{O}| = |G|$

$S = \{\text{vertices}\}$ has $|\mathcal{O}_s| = 6$, $|G_s| = 4 \Rightarrow |\mathcal{O}| = 6 \cdot 4 = 24$ ✓

$\{\text{edges}\}$ has $|\mathcal{O}_s| = 12$, $|G_s| = 2 \Rightarrow |\mathcal{O}| = 12 \cdot 2 = 24$ ✓

$\{\text{faces}\}$ has $|\mathcal{O}_s| = 8$, $|G_s| = 3 \Rightarrow |\mathcal{O}| = 8 \cdot 3 = 24$ ✓