§2.6. Isomorphisms

Def. Let G and H be groups. An *isomorphism* $G \to H$ is a homomorphism $G \to H$ which is bijective.

Ex: (a): $\exp : \mathbb{R}^+ \to \mathbb{R}^\times$ is not an isomorphism (since not surj.);

$\exp : \mathbb{R}^+ \to \mathbb{R}^\times_{>0}$ is an isomorphism.

(b) For any group G, and any $a \in G$, the map

$\mathbb{Z}^+ \to G, \quad n \mapsto a^n$

is an isomorphism $\underbrace{\text{iff}}_{\text{if \& only if}}$ $\underbrace{G = \langle a \rangle}_{\text{makes it surj.}}$ and $\underbrace{a \text{ has order } \infty}_{\text{makes it inj.}}$.

(c) $\mathrm{sign} : S_n \to \{\pm 1\}$ is an isomorphism iff $n = 2$.

Lemma 2.6.2. ~~A homom~~ The inverse of an isomorphism is an isomorphism.

Proof. Let φ be an isomorphism $(G \to H)$. Need to show: φ^{-1} is an isomorphism. Enough: φ^{-1} is a homomorphism.

- $\varphi^{-1}(xy) = \varphi^{-1}(x)\varphi^{-1}(y) \quad \forall x,y \in H$

(<u>Proof</u>: Write x and y as $\varphi(a)$ and $\varphi(b)$.

$$\Rightarrow \varphi^{-1}(xy) = \varphi^{-1}(\underbrace{\varphi(a)\varphi(b)}_{=\varphi(ab)}) = \varphi^{-1}(\varphi(ab))$$

$$= \underbrace{a}_{=\varphi^{-1}(x)}\underbrace{b}_{=\varphi^{-1}(y)} = \varphi^{-1}(x)\varphi^{-1}(y) \quad .)$$

$\square$

- Rest is similar.

<u>Def.</u> Two groups G, H are <u>isomorphic</u> if $\exists$ isomorphism $G \to H$.
Notation for this: $G \simeq H$ or $G \cong H$.

Intuitively clear: Isomorphic groups have "same properties",
Isomorphism acts as a dictionary between the groups.

<u>Def.</u> Let G be a group,
An <u>automorphism</u> of G means an isomorphism $G \to G$.
An <u>endomorphism</u> of G — // — homom — // — $G \to G$.

id_G is always an automorphism of G.

Def. Let G be a group and $g \in G$.
Let $conj_g : G \to G$ be the map sending each x to gxg^{-1}.
This is called "conjugation by g".
It is an automorphism of G.

[Proof: $\underbrace{conj_g(xy)}_{= gxyg^{-1}} = \underbrace{conj_g\, x}_{= gxg^{-1}} \cdot \underbrace{conj_g\, y}_{= gyg^{-1}}$? etc.

$= gx1yg^{-1}$

$g^{-1}(gxg^{-1})g = x$, where $g = (g^{-1})^{-1}$

$(conj_g)^{-1} = conj_{g^{-1}}$.]

Let $\operatorname{Aut} G$ be the set of all automorphisms of G.
$\operatorname{Aut} G$ is a group (subgroup of the symmetric group on G).

Finally,
$$G \longrightarrow \operatorname{Aut} G, \qquad g \longmapsto conj_g$$
is a group homomorphism.
(generally: neither injective nor surjective)

§2.7. Equivalence relations & set partitions

Def. Let S be a set.
A (set) <u>partition of S</u> is a set of disjoint nonempty subsets of S whose union is S.

Ex:
- $\{\{\text{even integers}\}, \{\text{odd integers}\}\}$ is a set partition of $\mathbb{Z}$.
- $\{\{1,2,5\}, \{3,4\}, \{6\}\}$ is a set partition of $\{1,2,\ldots,6\}$.
- $\{\{1\},\{2\}\} = \{\{2\},\{1\}\}$ is a set partition of $\{1,2\}$.
- $\{\text{species}\}$ is a set partition of $\{\text{organisms}\}$. $(\approx)$

Def. Let S be a set.
An <u>equivalence relation</u> (eqrel for short) is a binary relation $\sim$ on S with the following properties:

(a) <u>transitive:</u> If $a \sim b$ and $b \sim c$, then $a \sim c$.

(b) <u>symmetric:</u> If $a \sim b$, then $b \sim a$.

(c) <u>reflexive:</u> $a \sim a$ $\forall a \in S$.

EX:
- The relation "$=$" on any set S is an eqrel.
- The relation "$\cong$" on {groups} is an eqrel.
- Similarity of polygons — // —
- Given any group G, the relation of being conjugate is an eqrel on G. (Exercise.)
  ($a \sim b \iff \exists\, g \in G$ such that $a = g b g^{-1}$.)
- Similarity of matrices " is an eqrel.
- If S and T are two sets, and $f : S \to T$ is a map, ~~then~~ the relation "$\overset{f}{\equiv}$" on the set S defined by

$$\boxed{a \overset{f}{\equiv} b \iff f(a) = f(b)}$$

is an eqrel.

Def. If $\mathcal{P}$ is a set partition of a set S, then the elements of $\mathcal{P}$ are called the <u>blocks</u> or <u>parts</u> of $\mathcal{P}$. (They are subsets of S.)

Given $\mathcal{P}$, we can define an eqrel $\overset{\mathcal{P}}{\equiv}$ on S by

$a \overset{\mathcal{P}}{\equiv} b \iff$ a and b are in the same block of $\mathcal{P}$.

Ex: If $\mathcal{P} = \{\{1,2,5\}, \{3,4\}, \{6\}\}$, then

$1 \overset{\mathcal{P}}{\equiv} 2 \overset{\mathcal{P}}{\equiv} 5$, $\quad 3 \overset{\mathcal{P}}{\equiv} 4$, $\quad 6 \overset{\mathcal{P}}{\not\equiv} 3$ etc.

Def. Let $\sim$ be an eqrel. on S.
For any $s \in S$, let $C_s = \{b \in S \mid s \sim b\}$; this is called the equivalence class of s.
The set of all equivalence classes is a set partition of S, ~~(Artin calls it ...)~~ called $S/\sim$
(Artin calls it $\overline{S}$).

Prop. 2.7.4. Let S be a set. Then, the map

$\{\text{set partitions of } S\} \to \{\text{eqrels on } S\}$,

$\mathcal{P} \longmapsto \overset{\mathcal{P}}{\equiv}$

is a bijection; its inverse is

$$\{\text{eqrels on } S\} \longrightarrow \{\text{set partitions of } S\},$$
$$\sim \ \longmapsto \ S/\sim .$$

Ex: Let $\sim$ be the "conjugate" relation on S_3
(i.e., $a \sim b \iff a$ is conjugate to b).

What are the equiv. classes?

$C_1 = \{ g 1 g^{-1} \mid g \in S_3 \} = \{1\}.$

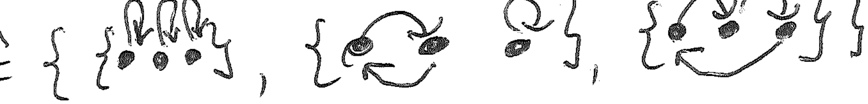
$C_{s_1} = \{ g s_1 g^{-1} \mid g \in S_3 \} = \{ s_1, s_2, s_1 s_2 s_1 \}.$

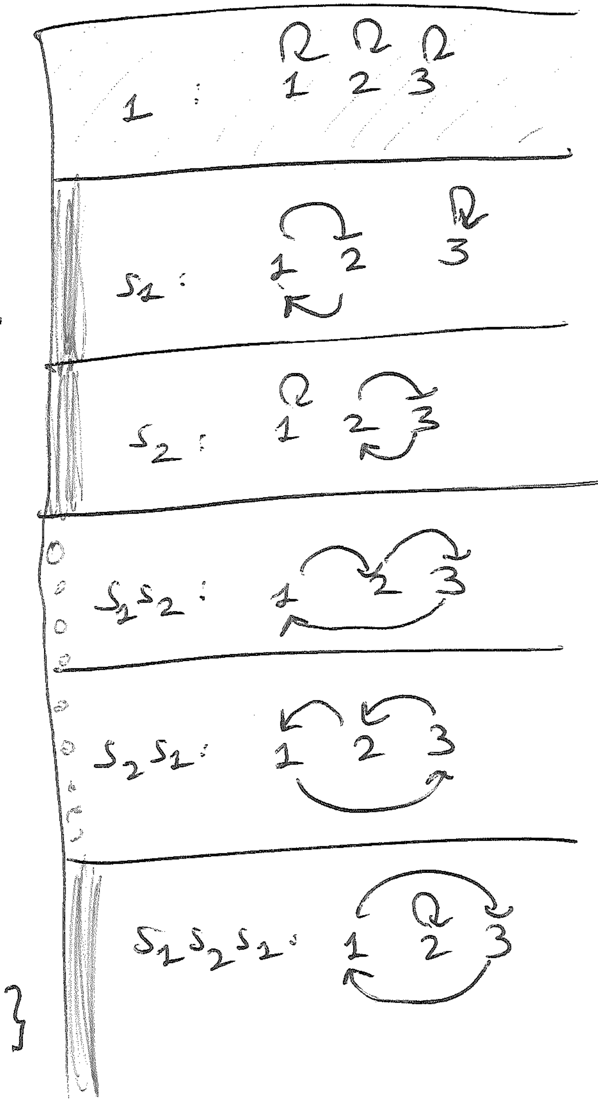
C_{s_2} = — // —

$C_{s_1 s_2}$ = $\{ s_1 s_2, s_2 s_1 \}.$

$C_{s_2 s_1}$ = — // —

$\Longrightarrow S_3/\sim \ = \{ \{1\}, \{ s_1, s_2, s_1 s_2 s_1 \},$
$\{ s_1 s_2, s_2 s_1 \} \}.$

$\hat{=}$ 



Rmk: If $f: S \to T$ is any map, then $\overset{f}{\equiv}$ is an eqrel on S (defined by $a \overset{f}{\equiv} b \iff f(a) = f(b)$).
This gives ~~a~~ set partition $S/\overset{f}{\equiv}$ of S.

Prop: Every set partition of S has the form $S/\overset{f}{\equiv}$ for some choice of T and f.

Proof. Let $\mathcal{P}$ be a set partition of S.
Let $\sim$ be the corresponding eqrel.
Let $\pi: S \longrightarrow \mathcal{P}$, $a \mapsto C_a$.
Then $\overset{\pi}{\equiv}$ is exactly $\sim$. Hence, $S/\overset{\pi}{\equiv} = S/\sim = \mathcal{P}$. $\square$