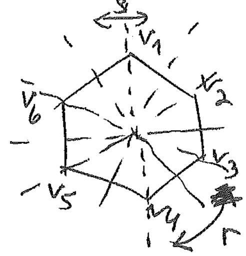
(75)

② $G = D_n$ = dihedral group acting on $S = \{\text{vertices } v_1, \ldots, v_n \text{ of } n\text{-gon}\}$

was transitive, and $\mathcal{O}_{v_1}$ has $G_{v_1} = \langle s \rangle = \{1, s\}$

$\mathcal{O}_{v_1} = S = \{v_1, \ldots, v_n\}$



$$|D_n| = |\mathcal{O}_{v_1}| \cdot |G_{v_1}|$$
$$2n = n \cdot 2$$

On the other hand, when $G = D_n$ acts on $S = \{\text{symmetry lines } \ell_0, \ell_1, \ldots, \ell_{n-1}\}$

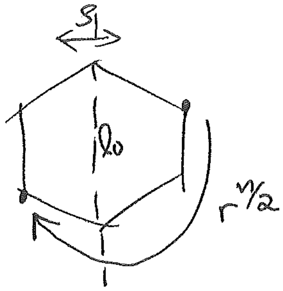
it was transitive for n even (because $G_{\ell_0} = \langle s \rangle$ again)

two orbits $\mathcal{O}_{\ell_0} = \mathcal{O}_{\ell_2} = \mathcal{O}_{\ell_4} = \ldots = \mathcal{O}_{\ell_{n-2}}$ for n odd

$\mathcal{O}_{\ell_1} = \mathcal{O}_{\ell_3} = \mathcal{O}_{\ell_5} = \ldots = \mathcal{O}_{\ell_{n-1}}$

(Q: What is G_{ℓ_0} for n even?)

$$G_{\ell_0} = \langle s, \underset{\text{rotation by } \pi}{r^{n/2}} \rangle \text{ if } n \text{ even}$$
$$= \{1, s, r^{n/2}, sr^{n/2}\}$$

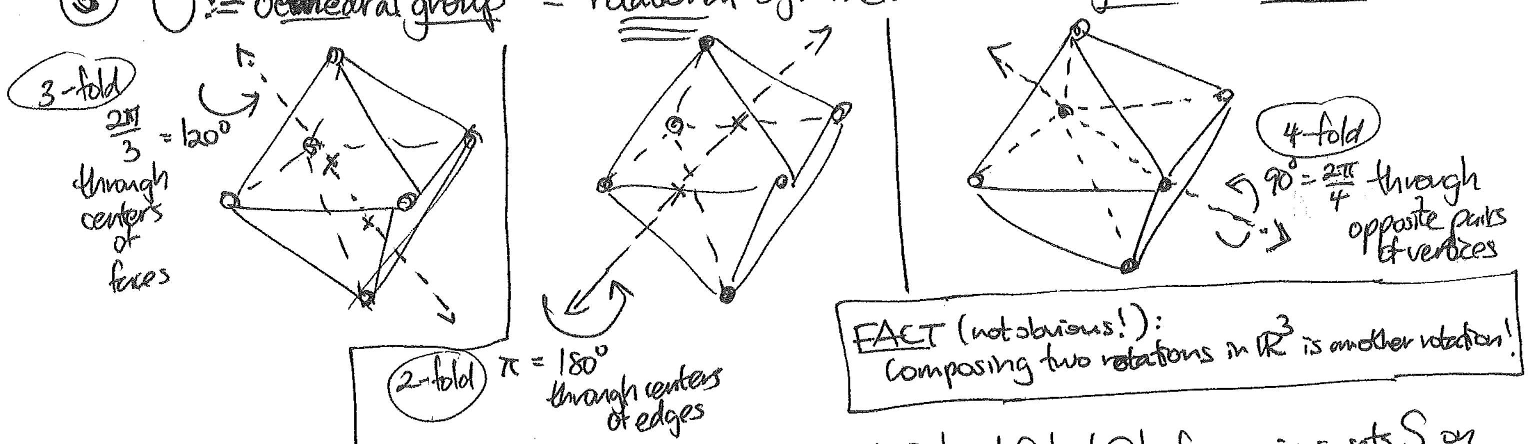


$$|D_n| = |\mathcal{O}_{\ell_0}| \cdot |G_{\ell_0}|$$

$$2n = \begin{cases} n \cdot 2 & \text{if } n \text{ odd,} \\ \frac{n}{2} \cdot 4 & \text{if } n \text{ even.} \end{cases}$$

10/22/2018 >

③ $G = O$:= octahedral group = (only!) rotational symmetries of a regular octahedron



FACT (not obvious!): Composing two rotations in $\mathbb{R}^3$ is another rotation!

What is the order of this group O?

$\underset{|G|}{|O|} = |\mathcal{O}_s| \cdot |G_s|$ for various sets S on which O acts like $S = \{\text{vertices}\}$, $S = \{\text{edges}\}$, $S = \{\text{faces}\}$

$S = \{\text{vertices}\}$ has $|\mathcal{O}_s| = 6$, $|G_s| = 4 \Rightarrow |O| = 6 \cdot 4 = 24$ ✓

$\{\text{edges}\}$ has $|\mathcal{O}_s| = 12$, $|G_s| = 2 \Rightarrow |O| = 12 \cdot 2 = 24$ ✓

$\{\text{faces}\}$ has $|\mathcal{O}_s| = 8$, $|G_s| = 3 \Rightarrow |O| = 8 \cdot 3 = 24$ ✓

 ... continuing 10/22/2018 >

§ 6.11 Permutation representations

DEF'N: When a group G acts on S, one says the action is faithful if $g*s=s \; \forall s\in S$ implies $g=1$.

One calls $K :=$ the kernel of the G-action on S

$$:= \{g\in G : g*s=s \; \forall s\in S\}$$
$$= \bigcap_{s\in S} G_s$$

so the action is faithful $\Leftrightarrow K=\{1\}$.

Why "kernel"?

PROPOSITION: A group action of G on X is the same as a homomorphism

$$G \xrightarrow{\varphi} S_X = \{\text{permutations } p: X\to X\}$$
$$g \longmapsto p_g = \begin{pmatrix} \cdots & x & \cdots \\ \cdots & g*x & \cdots \end{pmatrix}$$

In fact, $K = \bigcap_{x\in X} G_x = \ker\varphi$, so it is a normal subgroup of G.

Proof: Given a G-action, we know $p_{g^{-1}} = (p_g)^{-1}$ since $g*(g^{-1}*x) = (gg^{-1})*x = x$

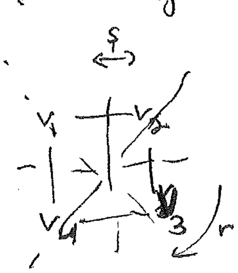
$$x \underset{g^{-1}}{\overset{g}{\rightleftarrows}} g*x$$

and φ is a homomorphism because

$$g'*(g*x) = (g'g)*x$$
$$\Rightarrow \underbrace{\varphi(g')\varphi(g)}_{p_{g'}p_g} = \underbrace{\varphi(g'g)}_{p_{g'g}}.$$

Conversely, given φ get the G-action on X from $g*x = \varphi(g)(x)$ ∎

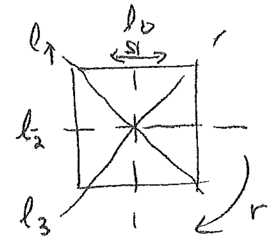
EXAMPLES: ① $G=D_4$ acting on $\{v_1,v_2,v_3,v_4\}$ = vertices of square



is faithful, so one gets an injective homomorphism $D_4 \xrightarrow{\varphi} S_{\{v_1,v_2,v_3,v_4\}} \cong S_4$

$$s \longmapsto (12)(34)$$
$$r \longmapsto (1234)$$

(2) However $G=D_4$ acting on $\{l_0, l_1, l_2, l_3\}$ = symmetry lines of the square



is not faithful,

since $r^2 = 180°$ rotation

lies in G_{l_i} $\forall i$, so $r^2 \in K = \bigcap_{i=0}^{3} G_{l_i} = \ker \varphi$

$$\text{where } D_4 \xrightarrow{\varphi} S_{\{l_0, l_1, l_2, l_3\}} \cong S_4$$
$$s \longmapsto (l_0)(l_1\ l_3)(l_2)$$
$$r \longmapsto (l_0\ l_2)(l_1\ l_3)$$
$$r^2 \longmapsto (l_0)(l_1)(l_2)(l_3)$$

(3) As a special case of G acting on left cosets $G/H = \{aH : a \in G\}$ via left-translation $g * aH = gaH$,

the $H = \{1\}$ case has G acting on G itself via $g * a = ga$.

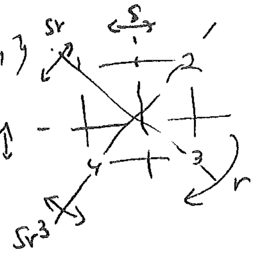
This is always faithful since
$$g * a = a$$
$$\Updownarrow$$
$$ga = a$$
$$\Updownarrow \quad \text{mult. by } a^{-1} \text{ on right}$$
$$g = 1$$

i.e. $G_a = \{1\}$ $\forall a \in G$, so $K = \bigcap_{a \in G} G_a = \{1\}$.

COR (Cayley): One has an injective homomorphism
(87.1)
$$G \xrightarrow{\varphi} S_G$$
for any group G, so if $|G| = n$ is finite,

$G \cong \operatorname{im} \varphi$, a subgroup of S_n.

EXAMPLE: $G = D_4$ acting on vertices $\{1,2,3,4\}$



gives a subgroup imφ of S_4 isomorphic to D_4, namely

$$\{1, \underset{r}{(1234)}, \underset{r^2}{(13)(24)}, \underset{r^3}{(1432)}, \underset{s}{(12)(34)}, \underset{sr}{(1)(3)(24)}, \underset{sr^2}{(14)(23)}, (13)(2)(4)\}$$

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§7.2 The class equation

Note that whenever a finite group G acts on a finite set S, if the G-orbits are $\mathcal{O}_{s_1}, \mathcal{O}_{s_2}, \ldots, \mathcal{O}_{s_t}$

then $S = \mathcal{O}_{s_1} \sqcup \mathcal{O}_{s_1} \sqcup \cdots \sqcup \mathcal{O}_{s_t}$

$$\Rightarrow |S| = |\mathcal{O}_{s_1}| + |\mathcal{O}_{s_1}| + \cdots + |\mathcal{O}_{s_t}| = \sum_{\text{orbits } \mathcal{O}_s} |\mathcal{O}_s|$$

and each $|\mathcal{O}_s| = \dfrac{|G|}{|G_s|}$, so

$$\boxed{|S| = \sum_{\text{orbits } \mathcal{O}_s} \frac{|G|}{|G_s|}}$$

NOTE: Every term $\frac{|G|}{|G_s|}$ divides into $|G|$!

10/24/2018 → In the special case where $S = G$ itself with G-action via conjugation
$$g * h := ghg^{-1}$$

then everything gets a special name:

$$\mathcal{O}_h = \{ghg^{-1} : g \in G\} = \text{the conjugacy class of } h \text{ in } G$$
$$=: C(h) \text{ or } C_G(h)$$

$$G_h = \{g \in G : ghg^{-1} = h \text{ i.e. } gh = hg\} = \text{the centralizer of } h \text{ in } G$$
$$=: Z(h) \text{ or } Z_G(h)$$

and
$$\boxed{|G| = \sum_{\substack{\text{conjugacy}\\ \text{classes } C(h)}} |C(h)| = \sum_{\substack{\text{conjugacy}\\ \text{classes } C(h)}} \frac{|G|}{|Z(h)|}}$$
is called the class equation of G