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§2.8 Cosets & Lagrange's Theorem

Recall for a subgroup $H < G$ a group,
a <u>left coset</u> is any set $aH := \{ah : h \in H\} \subset G$

<u>EXAMPLE:</u> For $H = \langle(12)\rangle = \{e, (12)\} < S_3 = G = \{e, (12), (13), (23), (123), (132)\}$

$H = eH = (12)H$ $\quad$ $(13)H = (123)H$ $\quad$ $(13)H = (132)H$

who are left cosets aH ?

$eH = \{e, (12)\} = H$

$(12)H = \{\underbrace{(12)e}_{=(12)}, \underbrace{(12)(12)}_{=e}\} = eH$

$h \mapsto (13)h$, $\quad (13)^{-1}g \leftarrow g$

$(13)H = \{\underbrace{(13)e}_{=(13)}, \underbrace{(13)(12)}_{=(123)}\}$

$(123)H = \{\underbrace{(123)e}_{=(123)}, \underbrace{(123)(12)}_{=(13)}\} = (13)H$

$h \mapsto (132)h$, $\quad (132)^{-1}g \leftarrow g$

$(23)H = \{\underbrace{(23)e}_{=(23)}, \underbrace{(23)(12)}_{=(132)}\}$

$(132)H = \{\underbrace{(132)e}_{=(132)}, \underbrace{(132)(12)}_{=(23)}\} = (23)H$

<u>PROPOSITION:</u> For $H < G$, defining $a \equiv b$ if $a = bh$ for some $h \in H$
(COR. 2.8.3, LEMMA 2.8.7) gives an equivalence relation on G whose equiv. classes are the (left) cosets $\{aH : a \in G\}$.

- Consequently, the left cosets $\{aH : a \in G\}$ partition G.
- Also, one always has a bijection
$$H \longrightarrow aH$$
$$h \longmapsto ah$$
$$(\text{with inverse } a^{-1}g \longleftarrow g),$$
so every coset aH has the same size, namely the order $|H|$ of H (possibly ∞).

<u>proof:</u> $\equiv$ is reflexive: $a \equiv a$ since $a = ae$ (and $e \in H$)
symmetric: $a \equiv b$, so $a = bh \Rightarrow b = ah^{-1}$ (and $h^{-1} \in H$)
transitive: $a \equiv b$, $b \equiv c$, so $\left.\begin{array}{l} a = bh_1 \\ b = ch_2 \end{array}\right\} \Rightarrow a = ch_2h_1$ (and $h_2h_1 \in H$)

exactly uses defn of subgroups!

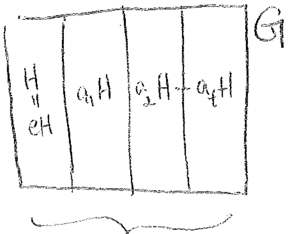
$\blacksquare$

This immediately gives...

<u>Lagrange's Theorem</u> (Thm 2.8.9) (+Cor 2.8.10): For a subgroup $H < G$ a <u>finite</u> group,

$$|G| = [G:H]\cdot|H|$$

where $\underline{[G:H]} := \#\{\text{left cosets } aH : a \in G\}$
the <u>index</u> of H in G



$t = \#$ of cosets aH $= [G:H]$

Hence $|H|$ <u>divides</u> $|G|$.

In particular, every $g \in G$ has its order $\mathrm{ord}(g)\ (:= |\langle g\rangle|)$ <u>dividing</u> $|G|$.

REMARK: $|G| = [G:H]\cdot|H|$ still is correct even if any of these numbers is ∞. Why?

<u>EXAMPLE</u>: In S_4,

p	$\mathrm{ord}(p)$
e	1
$(12), (13), \ldots$	2
$(12)(34), (13)(24), \ldots$	2
$(123), (124), \ldots$	3
$(1234), (1243), \ldots$	4

— all divide $|S_4| = 4! = 24$

S_4 does have <u>subgroups</u> H of orders $1, 2, 3, 4, 6, 8, 12, 24$ —
can you find one of each?
(perhaps 8 is trickiest)

<u>EXAMPLE</u> (COR 2.8.11) Every group G with $|G| = p$ a prime is <u>cyclic</u>,
and in fact $G = \langle g\rangle = \{1, g, g^2, \ldots, g^{p-1}\}$ for any $g \neq 1$ in G.

<u>proof</u>: One has $\{1\} \leq \langle g\rangle \leq G$

so $|\langle g\rangle|$ divides $|G| = p$ $\Rightarrow |\langle g\rangle| = 1$ or p since p is prime

$\Rightarrow |\langle g\rangle| = p$ since $g \neq 1$

$\Rightarrow \langle g\rangle = |G|$ ∎

An interesting situation comes from a group homomorphism $G \xrightarrow{\varphi} G'$
which has $\ker\varphi < G$, $\mathrm{im}\,\varphi < G'$ and the <u>fibers</u> of φ (which partition G) are exactly the cosets of $\ker\varphi$ i.e. $\varphi(a) = \varphi(b) \Leftrightarrow a\ker\varphi = b\ker\varphi$

<u>COROLLARY</u> (COR. 2.8.13) For a group homomorphism $G \xrightarrow{\varphi} G'$ with G, G' <u>finite</u>,
$|G| = |\ker\varphi|\cdot\underbrace{|\mathrm{im}\,\varphi|}_{=[G:\ker\varphi]}$, while $|\ker\varphi|$ divides $|G|$
and $|\mathrm{im}\,\varphi|$ divides both $|G|, |G'|$.

EXAMPLE: $S_n \xrightarrow{\text{sign}} \{\pm 1\}$ has $|\mathrm{im}(\text{sign})| = |\{\pm 1\}| = 2$

$p \longmapsto \text{sign}(p)$ and $\ker(\text{sign}) = A_n$ = alternating group = {even permutations in S_n}

hence $|A_n| \cdot \underbrace{2}_{[S_n:A_n]} = |S_n|$

so $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$

Note that here there are only two left cosets pA_n, namely A_n and pA_n where p is any odd permutation (sign$(p) = -1$).

The index $[G:H]$ behaves well in towers $K < H < G$ of subgroups:

follows anyway in finite case by canceling fractions: $\frac{|G|}{|K|} = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|}$

PROPOSITION: If $K < H < G$, then $[G:K] = [G:H] \cdot [H:K]$, even if one of these numbers is infinite.

In fact, if G has H-leftcosets $\{g_1H, g_2H, \ldots, g_mH\}$ where $m = [G:H]$ and H has K-leftcosets $\{h_1K, h_2K, \ldots, h_nK\}$ where $n = [H:K]$ then G has K-leftcosets $\{g_ih_jK\}_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$

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proof: Assume $g_1 = 1 = h_1$ W.L.O.G (without loss of generality).

Then $G = H \sqcup g_2H \sqcup \cdots \sqcup g_mH$ ($h \mapsto g_2h$ bijection)

$$= \begin{pmatrix} K \\ \sqcup h_1K \\ \sqcup h_2K \\ \vdots \\ \sqcup h_nK \end{pmatrix} \sqcup \begin{pmatrix} g_2K \\ \sqcup g_2h_1K \\ \sqcup g_2h_2K \\ \vdots \\ \sqcup g_2h_nK \end{pmatrix} \sqcup \begin{pmatrix} g_3K \\ \sqcup \\ \vdots \end{pmatrix} \sqcup \cdots \sqcup \begin{pmatrix} g_mK \\ \sqcup g_mh_1K \\ \sqcup g_mh_2K \\ \vdots \\ \sqcup g_mh_nK \end{pmatrix} \quad \blacksquare$$

EXAMPLE: A tower that we will come back to later inside S_4:

$$\{1\} \underset{\substack{[\langle(12)(34)\rangle:\{1\}] \\ =2}}{<} \langle (12)(34) \rangle \underset{\substack{[V_4:\langle(12)(34)\rangle] \\ =2}}{<} \{e, (12)(34), (13)(24), (14)(23)\} \underset{\substack{[A_4:V_4] \\ =3}}{<} A_4 \underset{\substack{[S_4:A_4] \\ =2}}{<} S_4$$

$\{e, (12)(34), (13)(24), (14)(23)\} = V_4$ Klein four $(\cong \{\begin{bmatrix} \pm1 & 0 \\ 0 & \pm1 \end{bmatrix}\} < GL_2(\mathbb{R}))$