

EXAMPLE: $S_n \xrightarrow{\text{sign}} \{\pm 1\}$ has $|\text{im}(\text{sign})| = |\{\pm 1\}| = 2$
 $p \mapsto \text{sign}(p)$ and $\ker(\text{sign}) = A_n = \text{alternating group} = \{\text{even permutations in } S_n\}$

hence $|A_n| \cdot 2 = |S_n|$
 $[S_n : A_n]$
 so $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$

Note that here there are only two ^{left} cosets pA_n , namely A_n and pA_n where p is any odd permutation ($\text{sign}(p) = -1$).

follows anyway in finite case by canceling fractions:
 $\frac{|G|}{|K|} = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|}$

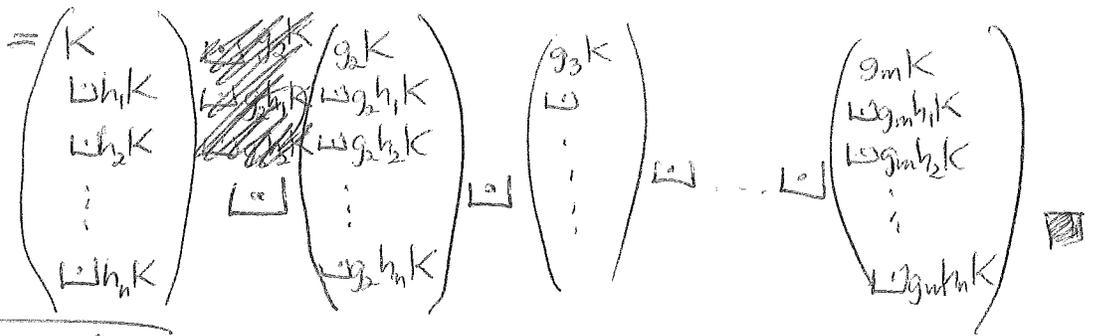
The index $[G:H]$ behaves well in towers $K < H < G$ of subgroups:

PROPOSITION: If $K < H < G$, then $[G:K] = [G:H] \cdot [H:K]$, even if one of these numbers is infinite.
 (PROP 2.8.14)

In fact, if G has H -leftcosets $\{g_1H, g_2H, \dots, g_mH\}$ where $m = [G:H]$ and H has K -leftcosets $\{h_1K, h_2K, \dots, h_nK\}$ where $n = [H:K]$ then G has K -leftcosets $\{g_i h_j K\}_{i=1, \dots, m; j=1, \dots, n}$

10/5/2018 > proof: Assume $g_i = 1 = h_1$ w.l.o.g. (without loss of generality).

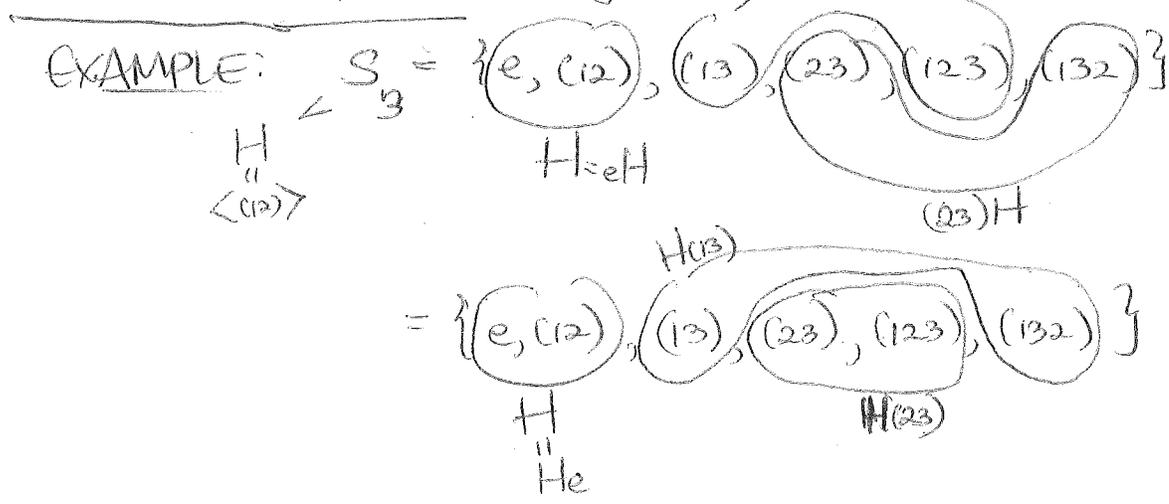
Then $G = H \sqcup g_2H \sqcup \dots \sqcup g_mH$



EXAMPLE: A tower that we will come back to later inside S_4 :

$\{1\} < \langle (12)(34) \rangle < \{e, (12)(34), (13)(24), (14)(23)\} < A_4 < S_4$
 $[K:1] = 1$ $[V_4:K] = 4$ $[A_4:V_4] = 3$ $[S_4:A_4] = 2$
 Klein four $V_4 \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \right\} < GL_2(\mathbb{R})$

We could have been working with right cosets $H_a := \{ha : a \in G\}$, which will be different in general



PROPOSITION: For any subgroup $H < G$, the sets $gHg^{-1} := \{ghg^{-1} : h \in H\}$ are all subgroups of G , isomorphic to H via the maps

(PROP 2-8.17, 2-8.18)

$$\begin{array}{ccc}
 H & \xrightarrow{\quad} & gHg^{-1} \\
 h & \xrightarrow{\quad} & ghg^{-1} \\
 \text{(with inverse } g^{-1}xg & \xleftarrow{\quad} & x)
 \end{array}$$

and TFAE for H :

- (i) $H \triangleleft G$ "H is a normal subgroup of G"
- i.e. $gHg^{-1} \subset H \quad \forall g \in G$
- (ii) $gHg^{-1} = H \quad \forall g \in G$
- (iii) $gH = Hg \quad \forall g \in G$
- (iv) Every left coset aH is also a right coset Hb
(and in fact, $aH = Ha = bH = Hb$ in this case).

Proof: (ii) \Leftrightarrow (iii) comes from multiplying on right by g or g^{-1} .

(ii) \Rightarrow (i) is clear, but then (i) \Rightarrow (ii) since $gHg^{-1} \subset H$
 $\Rightarrow H \subset g^{-1}Hg$ by $x \mapsto g^{-1}xg$
 $\Rightarrow H \subset g^{-1}Hg \subset H$
 i.e. $g^{-1}Hg = H$, so $gHg^{-1} = H$.

(iii) \Rightarrow (iv) is clear,
 but if every left coset is a right coset, then $gH = Hg'$ forces $g \in g'H = Hg'$
 $\Rightarrow Hg' = Hg$ i.e. $gH = Hg$

