(6)

More formally ...

<u>DEF'N</u>: $M \in \mathbb{R}^{m\times n}$ is in <u>row-echelon form</u> if

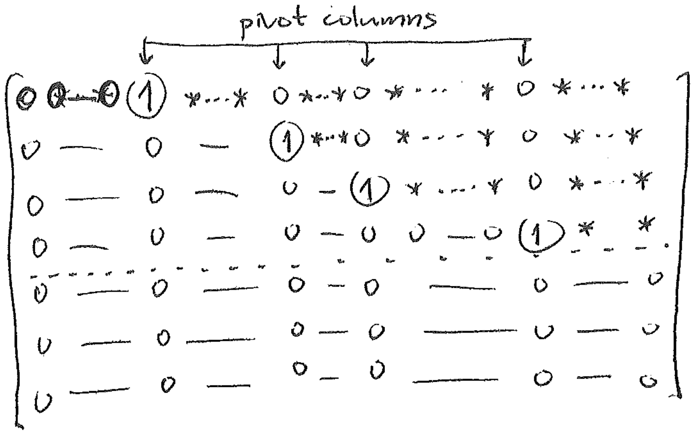
(a) its zero rows all come at the end ($\text{row } i \text{ zero} \Rightarrow \text{row } j \text{ is zero } \forall j \geq i$)

(b) each nonzero row has its leftmost nonzero entry being 1, (called a <u>pivot</u> 1)

(c) if rows i and $i+1$ are nonzero, the pivot 1 in row i is <u>left</u> of the pivot 1 in row $i+1$

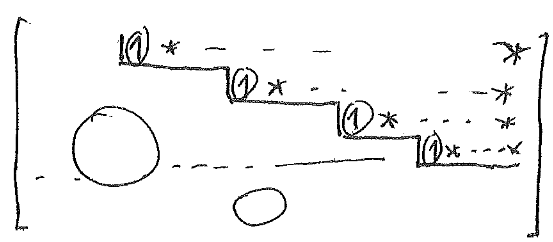
(d) pivot 1's are the only nonzero entry in their column.

e.g.



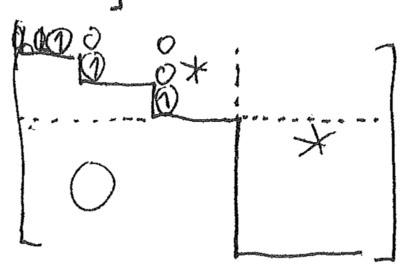
<u>PROPOSITION</u>: Every matrix $M \in \mathbb{R}^{m\times n}$ can be brought to row-echelon form by a sequence of row ops of type (i), (ii), (iii)

<u>proof</u>: Induct on the number of nonzero rows.
(sketch)



(<u>Q</u>: What does the <u>base case</u> of the induction look like?)

In the inductive step, one has it in this form



and one finds the <u>leftmost</u> nonzero entry below the dotted line, scales its row to make it a pivot 1, swaps rows until it is just below the dotted line, then uses the pivot 1 to eliminate entries in the same column. ∎

9/10/18

REMARK: The row-echelon form for M is <u>unique</u>, but we won't need this.

(7) Another key point is that the 3 types of row ops ~~are~~ can apply to M are the same as <u>multiplying M on the left</u> by 3 types of <u>elementary matrices</u>, all of which are invertible (with inverses also elementary matrices):

<u>Type (i)</u>:
$$E = \begin{array}{c} \\ \text{row } i \to \\ \\ \end{array} \begin{array}{c} \text{column } j \\ \downarrow \\ \begin{bmatrix} 1 & & \boxed{a} \quad 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \end{array}$$
has $M \mapsto EM$ adding a times row i of M to row j

and
$$E^{-1} = \begin{array}{c} \\ \text{row } i \to \\ \\ \end{array} \begin{array}{c} \text{column } j \\ \downarrow \\ \begin{bmatrix} 1 & & \boxed{-a} \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \end{array}$$

e.g. $j=1$, $i=2$:
$$\underbrace{\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}}_{M} = \begin{bmatrix} 5 & 6 & 7 \\ 8+5a & 9+6a & 10+7a \end{bmatrix}$$

<u>Type (ii)</u>:
$$E = \begin{array}{c} \\ \text{row } i \to \\ \\ \text{row } j \to \\ \\ \end{array} \begin{array}{c} \text{column } i \quad \text{column } j \\ \downarrow \qquad\quad \downarrow \\ \begin{bmatrix} 1 & & & & & & 0 \\ & 1 & & & & & \\ & & \boxed{0} & \cdots & \boxed{1} & & \\ & & \vdots & 1 & \vdots & & \\ & & \boxed{1} & \cdots & \boxed{0} & 1 & \\ 0 & & & & & & 1 \end{bmatrix} \end{array}$$
has $M \mapsto EM$ swapping rows i & j of M

and $E^{-1} = E$

e.g.
$$\begin{array}{c} \\ \text{row1} \to \\ \\ \text{row3} \to \end{array} \underbrace{\begin{array}{c} \text{row1} \quad \text{row3} \\ \downarrow \qquad \downarrow \\ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}}_{E} \underbrace{\begin{bmatrix} 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}}_{M} = \begin{bmatrix} 13 & 14 & 15 & 16 \\ 9 & 10 & 11 & 12 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

<u>Type (iii)</u>:
$$E = \begin{array}{c} \\ \text{row } i \to \\ \\ \end{array} \begin{array}{c} \text{column } i \\ \downarrow \\ \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \boxed{c} & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix} \end{array}$$
for $c \neq 0$

has $M \mapsto EM$ scaling row i of M by c

e.g.
$$\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 9c & 10c & 11c & 12c \end{bmatrix}$$

and
$$E^{-1} = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \boxed{\tfrac{1}{c}} & & \\ & & & 1 & \\ 0 & & & & 1 \end{bmatrix}$$

This will have a lot of consequences, starting with a simple observation.

<u>PROP</u>: If $A_1, A_2, \ldots, A_r$ have (2-sided) inverses $A_1^{-1}, \ldots, A_r^{-1}$ then so does their <u>product</u> $A_1 A_2 \cdots A_r$, namely $(A_1 A_2 \cdots A_r)^{-1} = A_r^{-1} A_{r-1}^{-1} \cdots A_2^{-1} A_1^{-1}$

Q: Does this make sense <u>without</u> <u>parentheses</u>? E.g. is $A(B(CD)) = ((AB)C)D$?

$$A(B(CD)) = A(BC)D = (A(BC))D = ((AB)C)D$$

In particular, a product $E_1 E_2 \cdots E_r$ of <u>elementary</u> matrices is always invertible.

<u>Proof</u>: Check $A_1 A_2 \cdots A_r \cdot A_r^{-1} A_{r-1}^{-1} \cdots A_2^{-1} A_1^{-1} = I$ and $A_r^{-1} \cdots A_2^{-1} A_1^{-1} \cdot A_1 A_2 \cdots A_r = I$ by induction on r ■

(8) <u>COROLLARY</u>:
(THM 1.2.16, PROP 1.2.20)

For a matrix $A \in \mathbb{R}^{m \times n}$,

(i) A can never have a right-inverse $AR = I_m$ if $m > n$
(but <u>might</u> have a left-inverse)

(ii) A — " — left-inverse $LA = I_n$ if $m < n$
(but <u>might</u> have a right-inverse)

(iii) if $m = n$ so A is square $n \times n$, $\underbrace{\text{the following are equivalent}}_{\text{(T.F.A.E.)}}$:

(a) A has I_n as a row-echelon form

(b) $A = E_1 E_2 \cdots E_r$ for some elementary matrices E_i

(c) A has a (two-sided) <u>inverse</u>

(d) A has a right-inverse.

(e) A has a left-inverse

proof: <u>(i)</u>: If $m > n$ then the echelon form A' for A definitely has some zero rows:

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix} \xrightarrow{\text{row-reduce}} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix} = A' = \overbrace{E_1 E_2 \cdots E_r}^{\text{elementary matrices}} A$$
$$= PA$$

where $P = E_1 E_2 \cdots E_r$ is <u>invertible</u>.

Then if A had a right-inverse R,

$$AR = I_m$$
$$\Downarrow$$
$$P(AR)P^{-1} = P I_m P^{-1} = \underbrace{I_m}_{\text{has no zero row}}$$
$$\parallel$$
$$\underbrace{A'}_{\text{has a zero row}} \cdot RP^{-1}$$

$\boxed{\text{Contradiction}}$

(9)

(ii) If $m<n$ and A had a left-inverse $LA = I_n$ (with $L \in \mathbb{R}^{n\times m}$),

then applying (i) to L gives a contradiction, since A is a right-inverse to L.

(iii) Let's prove $(d) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c)$ first, assuming $A \in \mathbb{R}^{n\times n}$.

(d) ⇒ (a): If the echelon form A' for A has $A' \neq I_n$,

then A' has a zero row:
$$A' = \begin{array}{c} \\ n\left\{\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ \hline 0 & \cdots & \cdots & 0 \end{bmatrix}\right. \\ \quad\; n \end{array}$$

Then as above in (i), A cannot have a right inverse $AR = I_n$

or else since $A' = PA$ for some invertible P, one has

$$\underset{\underbrace{A'}_{\text{has a zero row}}(RP^{-1})}{PARP^{-1}} = PI_nP^{-1} = \underbrace{I_n}_{\text{has no zero row}}$$

$\boxed{\text{contradiction}}$

(a) ⇒ (b): If $\underset{\substack{\text{echelon} \\ \text{form for } A}}{A'} = I_n = E_r E_{r-1} \cdots E_2 E_1 A$

then multiplying by $E_1^{-1} E_2^{-1} \cdots E_r^{-1}$ gives $\underbrace{E_1^{-1} E_2^{-1} \cdots E_r^{-1}}_{\text{a product of elementary matrices}} = A$

(b) ⇒ (c): $A = E_1 \cdots E_r \Rightarrow A^{-1} = E_r^{-1} \cdots E_1^{-1}$ as shown earlier,
is a 2-sided inverse for A.

(c) ⇒ (d): Obviously 2-sided inverses are right inverses.

Now let's check (e) ⇒ (c), since (c) ⇒ (e) is also obvious.

If A has a left-inverse $LA = I_n$, with $L \in \mathbb{R}^{n\times n}$

the L has A as a right-inverse, so L has a 2-sided inverse B by

what we've shown already, but then $\underset{B}{B(LA)} = \underset{A}{(BL)A}$ ■