9/12/18 >

(10)

Row-reduction also answers our linear system questions...

THEOREM: Given a linear system $AX=Y$ with $M:=[A|Y]$ (so if $A\in\mathbb{R}^{m\times n}$ then $M\in A^{m\times(n+1)}$) having $M'=[A'|Y']$ as a row-echelon form,

(PROP 1.2.13, plus a lot more)

(i) $\left\{\begin{matrix}\text{solutions } X \text{ to}\\ AX=Y\end{matrix}\right\} = \left\{\begin{matrix}\text{solutions } X \text{ to}\\ A'X=Y'\end{matrix}\right\}$

(ii) Such solutions X exist $\iff$ $M'=[A'|Y']$ has <u>no pivot 1's in its last column Y'</u>

(iii) If such solutions X exist, either

(a) they're <u>unique</u> if every column of A' contains a pivot 1,

(b) otherwise for each nonpivot column i of A' one can pick $x_i\in\mathbb{R}$ <u>arbitrarily</u>, and this determines the x_j for the pivot columns to get all solutions X.

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e.g.

$$M=\underbrace{\left[\begin{matrix}-2&-6&-5&-11\\1&3&4&4\\1&3&6&2\end{matrix}\right.}_{A}\underbrace{\left|\begin{matrix}1\\1\\3\end{matrix}\right]}_{Y} \quad \xrightarrow{\text{row-reduction}} \quad M'=\begin{array}{c}\begin{matrix}x_1&x_2&x_3&x_4&\ \end{matrix}\\ \underbrace{\left[\begin{matrix}1&3&0&8\\0&0&1&-1\\0&0&0&0\end{matrix}\right.}_{A'}\underbrace{\left|\begin{matrix}-3\\1\\0\end{matrix}\right]}_{Y'}\end{array}$$

had solutions:

$$X=\begin{bmatrix}1+x_4\\x_2\\-3-3x_2-8x_4\\x_4\end{bmatrix}$$

arbitrary

whereas if instead

$$M=\underbrace{\left[\begin{matrix}-2&-6&-5&-11\\1&3&4&4\\1&3&6&2\end{matrix}\right.}_{A}\underbrace{\left|\begin{matrix}1\\1\\5\end{matrix}\right]}_{Y} \quad \xrightarrow{\text{row reduction}} \quad M'=\begin{array}{c}\begin{matrix}x_1&x_2&x_3&x_4&\ \end{matrix}\\ \underbrace{\left[\begin{matrix}1&3&0&8\\0&0&1&-1\\0&0&0&0\end{matrix}\right.}_{A'}\underbrace{\left|\begin{matrix}0\\0\\1\end{matrix}\right]}_{Y'}\end{array} \Rightarrow \text{no solutions } X$$

$\boxed{\text{pivot 1 in } Y'}$

(11)

proof of THM:

(i): Since M' is the row reduction of M, $M' = E_1E_2\cdots E_rM$ with E_i elementary
$= PM$ with $P = E_1E_2\cdots E_r$ invertible.

$$M' = PM$$

Hence $[A'|Y'] = P[A|Y] = [PA|PY]$

meaning $A' = PA$, $Y' = PY$, and one has

$$AX = Y$$

$$\Big\Downarrow \text{ mult. on left by } P \qquad \Big\Uparrow \text{ mult. on left by } P^{-1}$$

$$PAX = PY$$

i.e. $A'X = Y'$

(ii) and (iii): If $M' = [A'|Y']$ has a pivot 1 in Y'

then the system $A'X = Y'$ has an equation $0 = 1$,

so no solutions.

Otherwise all the pivot 1's are in columns of A', and one can see how to get all solutions from arbitrarily assigning the $x_j \in \mathbb{R}$ for non-pivot columns j of A':

$$\begin{array}{ccccccccccc}
x_1\ x_2 & x_{p_1} & & x_{p_2} & & x_{p_3} & & x_{p_r} & x_n & & \\
0 \cdots 0 & 1 & * \cdots * & 0 & * \cdots * & 0 & * \cdots * & 0 & * \cdots * & \big| & * \\
 & 0 & & 1 & * \cdots * & 0 & * \cdots * & 0 & * \cdots * & \big| & * \\
 & 0 & & 0 & & 1 & * \cdots * & 0 & * \cdots * & \big| & * \\
 & 0 & & 0 & & 0 & & 1 & * \cdots * & \big| & * \\
 & 0 & & 0 & & & & & & \big| & 0 \\
 & & & & 0 & & & & & & \\
 & & & & A' & & & & & & Y'
\end{array}$$

$\leftarrow$ $x_{p_1}, x_{p_2}, \ldots, x_{p_r}$ are determined by the non-pivot x_i's.

COROLLARY: For square $A \in \mathbb{R}^{n\times n}$, T.F.A.E.
(THM 1.2.21)

(a) A invertible

(b) $AX = Y$ has a unique solution X for any choice of Y

(c) $AX = 0_n$ has only the solution $X = 0_n := \left.\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} n$.

proof: (a) $\Rightarrow$ (b) since A invertible $\Rightarrow$ $AX = Y$ forces $A^{-1}AX = A^{-1}Y$, $X = A^{-1}Y$

(b) $\Rightarrow$ (c) trivially since $X = 0_n$ is at least one solution to $AX = 0_n$

(c) $\Rightarrow$ (a) since $AX = 0$ having only one solution $\Rightarrow$ A has echelon form A' with all pivot columns in A', so $A' = I_n$ ∎

§1.3 Transpose

DEF'N: $A \in \mathbb{R}^{m\times n}$ has transpose matrix $A^t \in \mathbb{R}^{n\times m}$

$$A = (a_{ij})_{\substack{i=1,\dots,m\\ j=1,\dots,n}} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad A^t = (a_{ji})_{\substack{j=1,\dots,n\\ i=1,\dots,m}} = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}$$

that is, $(A^t)_{ij} := A_{ji}$

Not hard to check things like

$$(A^t)^t = A$$
$$(A+B)^t = A^t + B^t$$
$$(cA)^t = c(A^t)$$
$$\boxed{(AB)^t = B^t A^t}$$ ← worth trying as an easy exercise

$$\Rightarrow \boxed{(A^t)^{-1} = (A^{-1})^t \text{ if } A \text{ invertible}}$$

Because of $A \mapsto A^t$, many statements about row ops/reduction, have analogues for col ops/reduction.

we'll come back to §1.4...

§ 1.5 Permutations and their signs

DEF'N: $S_n = \{ \text{all } \underline{\text{permutations}} \ \{1,2,..,n\} \xrightarrow{p} \{1,2,..,n\} \}$

$= \underline{\text{bijections}}$

$= \text{injective and surjective functions}$

is called the symmetric group (on n letters)

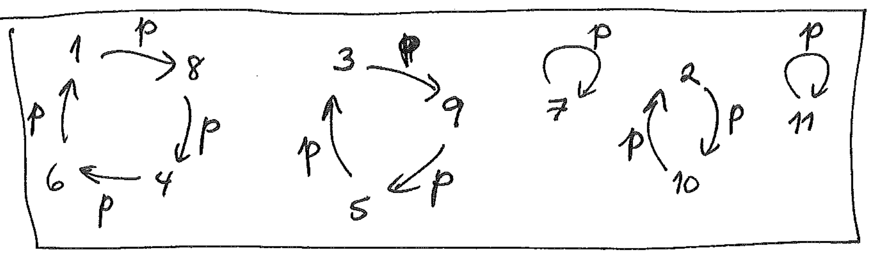
recall injective means $p(x)=p(y) \Rightarrow x=y$
surjective means $\forall y \ \exists x$ with $p(x)=y$

We'll use several notations for them!

EXAMPLE: In S_{11}, $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 8 & 10 & 9 & 6 & 3 & 1 & 7 & 4 & 5 & 2 & 11 \end{pmatrix}$ in two-line notation

meaning
$p(1)=8$
$p(2)=10$
$\vdots$
$p(11)=11$

can also be written in functional directed graph notation as



or in cycle notation(s) $p = (1\,8\,4\,6)(3\,9\,5)(7)(2\,10)\ (11)$

$= (4\,6\,1\,8)(9\,5\,3)(7)(10\ 2)\ (11)$

$= (4\,6\,1\,8)(9\,5\,3)(10\ 2)$

$= (7)(11)(2\,10)(3\,5\,9)(6\,1\,8\,4)$

$\vdots$ etc.

Permutations can be composed, and order matters!

e.g. $p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = (1\,2\,3)(4) = (123)$

$q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix} = (14)(2)(3) = (14)$

have $pq = (123)(14) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} = (1423)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \#$

$qp = (14)(123) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = (1234)$