This is consistent with multiplication of matrices through ..

DEF'N: Given $p = \begin{pmatrix} 1 & 2 & \cdots & n \\ p(1) & p(2) & \cdots & p(n) \end{pmatrix} \in S_n$

its associated permutation matrix $P \in \mathbb{R}^{n \times n}$

has exactly one nonzero entry in each row and column,

namely $P_{ij} = 1$ whenever $p(j) = i$

$$\begin{array}{cc} & \text{col } j \\ \text{row } i = p(j) & P = \begin{bmatrix} & & & 0 & & \\ & & & \vdots & & \\ & & & 0 & & \\ 0 & \cdots & 0 & 1 & 0 & \cdots \\ & & & 0 & & \\ & & & 0 & & \end{bmatrix} \end{array}$$

e.g. $p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \leftrightarrow P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix} \leftrightarrow Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$pq = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} \longleftrightarrow PQ = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ ✓

In other words,

$p(j) = i \iff Pe_j = e_i$

where

e_j = j^{th} standard basis vector $= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th}$

Permutations $p \in S_n$ always have a (2-sided) inverse $p^{-1} \in S_n$

meaning $p \cdot p^{-1} = e := \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$
the identity permutation in S_n

that one gets by turning its two-line notation upside down,

or reversing the arrows in its functional directed graph,

or reading each cycle backwards

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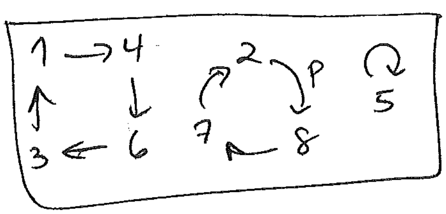
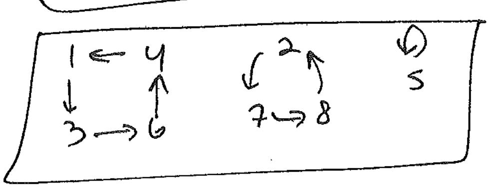
e.g. $p =$ 

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 8 & 1 & 6 & 5 & 3 & 2 & 7 \end{pmatrix} = (1463)(287)(5)$

has $p^{-1} =$ 

$= \begin{pmatrix} 4 & 8 & 1 & 6 & 5 & 3 & 2 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = (3641)(782)(5)$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 6 & 1 & 5 & 4 & 8 & 2 \end{pmatrix}$

(15) <u>DEFINITION</u> - <u>PROPOSITION</u> (!):

There is an (interesting) function $S_n \xrightarrow{\text{sign}} \{+1,-1\}$

$$p \longmapsto \text{sign}(p) := \prod_{1\le i<j\le n} \frac{p(j)-p(i)}{j-i}$$

having these properties:

(i) $\text{sign}(p) = (-1)^{\#\{(i,j):\ 1\le i<j\le n,\ p(i)>p(j)\}}$

called an <u>inversion</u> pair for p

(ii) $\text{sign}(\underset{\left(\begin{smallmatrix}1&2&\cdots&n\\1&2&\cdots&n\end{smallmatrix}\right)}{\overset{}{e}}) = +1$

(iii) $\text{sign}((k\ \ell)) = -1$ $\forall$ <u>transpositions</u> $(k\ \ell)$, $1\le k<\ell\le n$

(iv) $\text{sign}(pq) = \text{sign}(p)\,\text{sign}(q)$

REMARK: $S_n \xrightarrow{\text{sign}} \{\pm 1\}$ will be (later) our 1st interesting example of a <u>group homomorphism</u>

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e.g. $\text{sign}(\underset{\left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right)}{\overset{p}{(1\,3)}}) = \prod_{1\le i<j\le 3} \frac{p(j)-p(i)}{j-i} = \frac{(2-3)(1-3)(1-2)}{(2-1)(3-1)(3-2)} = \frac{1-2}{2-1}\cdot\frac{1-3}{3-1}\cdot\frac{2-3}{3-2} = (-1)(-1)(-1)$
$= (-1)^3 = -1$

Say (13) is an <u>odd</u> permutation since $\text{sign}((13)) = -1$

$\text{sign}\left(\left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)\right) = \frac{(3-2)(1-2)(1-3)}{(2-1)(3-1)(3-2)} = \frac{1-2}{2-1}\cdot\frac{1-3}{3-1}\cdot\frac{3-2}{3-2} = (-1)(-1)(+1) = (-1)^2 = +1$

Say $\left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)$ is an <u>even</u> permutation $\text{sign}\left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right) = +1$

<u>proof of DEF'N - PROP</u>:

First note $\text{sign}(p) = \prod_{i<j} \frac{p(j)-p(i)}{j-i}$ really does lie in $\{\pm 1\}$,

since for each denominator factor $j-i$ one can find a <u>unique</u> occurrence of either $j-i$ or $i-j$ in the numerator,

and $\frac{j-i}{j-i} = +1$, $\frac{i-j}{j-i} = -1$.

Then (i) follows because the denominator is all <u>positive</u> factors $j-i$ with $1\le i<j\le n$, while the <u>numerator</u> has exactly as many negative factors $p(j)-p(i)$ as $\#\{(i,j): 1\le i<j\le n,\ p(i)>p(j)\}$, so this predicts its sign.

Assertion (ii) follows since $\mathrm{sign}(e) = \prod_{i<j} \frac{j-i}{j-i} = +1$ (or since $e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$ has zero inversion pairs (i,j)).

Assertion (iii) follows from computing

$$\mathrm{sign}((k\ l)) = \mathrm{sign}\left(\left(\begin{array}{ccc|c|ccc|c|ccc} 1 & 2 \cdots & k-1 & k & k+1 & \cdots & l-1 & l & l+1 & \cdots & n \\ 1 & 2 \cdots & k-1 & l & k+1 & \cdots & l-1 & k & l+1 & \cdots & n \end{array} \right) \right)$$

$$= (-1)^{\underbrace{(l-k-1)}_{\substack{\text{inversions between} \\ l \text{ and } k+1,\ldots,l-1}} + \underbrace{(l-k-1)}_{\substack{\text{inversions between} \\ k+1,\ldots,l-1 \text{ and } k}} + \underbrace{1}_{\substack{\text{the inversion} \\ (l,k)}}}$$

$$= +1$$

For assertion (iv),

$$\mathrm{sign}(pq) = \prod_{i<j} \frac{p(q(j)) - p(q(i))}{j-i} = \prod_{i<j} \frac{p(q(j)) - p(q(i))}{q(j) - q(i)} \cdot \underbrace{\prod_{i<j} \frac{q(j) - q(i)}{j-i}}_{\mathrm{sign}(q)}$$

$$\prod_{i<j} \frac{p(q(j)) - p(q(i))}{q(j) - q(i)} = \underbrace{\prod_{i'<j'} \frac{p(j') - p(i')}{j' - i'}}_{\mathrm{sign}(p)}$$

Let $i' = q(i)$, $j' = q(j)$ if $q(i) < q(j)$

otherwise

let $i' = q(j)$, $j' = q(i)$ if $q(i) > q(j)$

$\blacksquare$

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§1.6, 1.4 Determinants - an important theoretical tool, less useful computationally

DEF'N: The determinant function $\mathbb{R}^{n \times n} \xrightarrow{\det} \mathbb{R}$

sends $A = (a_{ij})_{\substack{i=1,\ldots,n \\ j=1,\ldots,n}} \mapsto \det A := \sum_{p \in S_n} \mathrm{sign}(p)\, a_{1,p(1)} a_{2,p(2)} \cdots a_{n,p(n)}$

e.g. $\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \underbrace{+ a_{11} a_{22}}_{p = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}} \underbrace{- a_{12} a_{21}}_{p = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}$

$\begin{bmatrix} a_{11} & \cdot \\ \cdot & a_{22} \end{bmatrix}$ $\begin{bmatrix} \cdot & a_{12} \\ a_{21} & \cdot \end{bmatrix}$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix} = \underset{p=\binom{1\,2\,3}{1\,2\,3}}{+a_{11}a_{22}a_{33}} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} + \underset{p=\binom{1\,2\,3}{2\,3\,1}}{a_{12}a_{23}a_{31}}$$

$$\begin{bmatrix}\circ & \cdot & \cdot \\ \cdot & \circ & \cdot \\ \cdot & \cdot & \circ\end{bmatrix} \quad \begin{bmatrix}\cdot & \circ & \cdot \\ \circ & \cdot & \cdot \\ \cdot & \cdot & \circ\end{bmatrix} \quad \begin{bmatrix}\circ & \cdot & \cdot \\ \cdot & \cdot & \circ \\ \cdot & \circ & \cdot\end{bmatrix} \quad \begin{bmatrix}\cdot & \cdot & \circ \\ \cdot & \circ & \cdot \\ \circ & \cdot & \cdot\end{bmatrix} \quad \begin{bmatrix}\cdot & \cdot & \circ \\ \circ & \cdot & \cdot \\ \cdot & \circ & \cdot\end{bmatrix} \quad \begin{bmatrix}\cdot & \circ & \cdot \\ \cdot & \cdot & \circ \\ \circ & \cdot & \cdot\end{bmatrix}$$

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Determinants have <u>many</u> properties ...

<u>THEOREM:</u> Regarded as a function $\det(A) = \det(A_1, \ldots, A_n)$ of the n column vectors $A_1, \ldots, A_n$ of A,

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$

(i) det is <u>linear</u> in each column, meaning

$$\det(A_1, \ldots, A_i + A_i', \ldots, A_n) = \det(A_1, \ldots, A_i, \ldots, A_n) + \det(A_1, \ldots, A_i', \ldots, A_n)$$

<u>and</u> $\det(A_1, \ldots, cA_i, \ldots, A_n) = c\det(A_1, \ldots, A_n)$ for any $c \in \mathbb{R}$

(ii) det is <u>alternating</u> in the columns, meaning

$$\det(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n) = -\det(A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n)$$

(iii) $\det(I_n) = +1$.

Furthermore, properties (i), (ii), (iii) (uniquely) completely <u>characterize</u> $\mathbb{R}^{n\times n} \xrightarrow{\det} \mathbb{R}$.

<u>proof:</u> Property (i) holds for each of the $n!$ simpler functions $A \longmapsto \operatorname{sign}(p)\, a_{1p(1)} a_{2p(2)} \cdots a_{np(n)}$

$$= \pm\begin{bmatrix} & & \circ & & \\ \circ & & & & \circ \\ & & \circ & & \\ & & & \circ & \end{bmatrix}$$

and hence it holds for det, which you get by summing these $n!$ simpler functions.

Property (ii) holds for det because a term $\operatorname{sign}(p)\, a_{1p(1)} \cdots a_{np(n)}$ in $\det(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n)$ will correspond to a term $\operatorname{sign}(p')\, a_{1p'(1)} \cdots a_{np'(n)}$ in $\det(A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n)$ in which $p' = p\cdot(ij)$

so $\operatorname{sign}(p') = \operatorname{sign}(p\cdot(ij)) = \operatorname{sign}(p)\cdot\operatorname{sign}(ij) = -\operatorname{sign}(p)$