(17)

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix} = \underset{p=\binom{1\,2\,3}{1\,2\,3}}{+a_{11}a_{22}a_{33}} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} + \underset{p=\binom{1\,2\,3}{2\,3\,1}}{a_{12}a_{23}a_{31}}$$

$$\begin{bmatrix} \circ & \cdot & \cdot \\ \cdot & \circ & \cdot \\ \cdot & \cdot & \circ \end{bmatrix} \quad \begin{bmatrix} \cdot & \circ & \cdot \\ \circ & \cdot & \cdot \\ \cdot & \cdot & \circ \end{bmatrix} \quad \begin{bmatrix} \circ & \cdot & \cdot \\ \cdot & \cdot & \circ \\ \cdot & \circ & \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \circ \\ \cdot & \circ & \cdot \\ \circ & \cdot & \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot & \cdot & \circ \\ \circ & \cdot & \cdot \\ \cdot & \circ & \cdot \end{bmatrix} \quad \begin{bmatrix} \cdot & \circ & \cdot \\ \cdot & \cdot & \circ \\ \circ & \cdot & \cdot \end{bmatrix}$$

9/17/18 >

Determinants have <u>many</u> properties ...

<u>THEOREM</u>: Regarded as a function $\det(A) = \det(A_1, \ldots, A_n)$ of the n column vectors $A_1, \ldots, A_n$ of A,

$$A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix}$$

(i) det is ~~linear~~ in each column, meaning

$$\det(A_1, \ldots, A_i + A_i', \ldots, A_n) = \det(A_1, \ldots, A_i, \ldots, A_n) + \det(A_1, \ldots, A_i', \ldots, A_n)$$

<u>and</u> $\det(A_1, \ldots, cA_i, \ldots, A_n) = c\det(A_1, \ldots, A_n)$ for any $c \in \mathbb{R}$

(ii) det is <u>alternating</u> in the columns, meaning

$$\det(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n) = -\det(A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n)$$

(iii) $\det(I_n) = +1$.

Furthermore, properties (i), (ii), (iii) (uniquely) completely <u>characterize</u> $\mathbb{R}^{n\times n} \xrightarrow{\det} \mathbb{R}$.

<u>proof:</u> Property (i) holds for each of the $n!$ simpler functions $A \longmapsto \operatorname{sign}(p)\, a_{1p(1)} a_{2p(2)} \cdots a_{np(n)}$

and hence it holds for det, which you get by summing these $n!$ simpler functions.

Property (ii) holds for det because a term $\operatorname{sign}(p)\, a_{1p(1)} \cdots a_{np(n)}$ in $\det(A_1, \ldots, A_i, \ldots, A_j, \ldots, A_n)$ will correspond to a term $\operatorname{sign}(p')\, a_{1p'(1)} \cdots a_{np'(n)}$ in $\det(A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n)$ in which $p' = p \cdot (ij)$

so $\operatorname{sign}(p') = \operatorname{sign}(p \cdot (ij)) = \operatorname{sign}(p) \cdot \operatorname{sign}(ij) = -\operatorname{sign}(p)$

(18)

Property (iii) holds for det since the only nonzero term in

$$\det(I_n) = \det\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \text{ is the } p = e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} \text{ term } +a_{11}a_{22}\cdots a_{nn} = +1$$

To see that (i), (ii), (iii) characterize $\mathbb{R}^{n\times n} \xrightarrow{\det} \mathbb{R}$, assume one has some function $\mathbb{R}^{n\times n} \xrightarrow{\delta} \mathbb{R}$ satisfying (i), (ii), (iii), and we'll show $\delta(A)$ is uniquely calculable using (i), (ii), (iii).

Note that for elementary matrices E, one has

$$\delta\begin{bmatrix} 1 & & \boxed{a} \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix} \underset{\text{by (i)}}{=} \delta\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{bmatrix} + \delta\begin{bmatrix} 1 & & \boxed{a} \\ & \ddots & \\ 0 & & \boxed{0} \\ & & & 1 \end{bmatrix}$$

$$\underset{\substack{\text{by (iii)}\\\text{and (i)}}}{=} 1 + a\cdot\underbrace{\delta\begin{bmatrix} \boxed{1} & \cdots & \boxed{1} \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \end{bmatrix}}_{} \underset{\text{by (ii)}}{=} 1$$

has two columns equal, so its δ value is the same as its negative, i.e. zero by (ii)

$$\delta\begin{bmatrix} 1 & & & & \\ & 1 & & & 0 \\ & & \boxed{\begin{matrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{matrix}} & & \\ 0 & & & 1 & \\ & & & & 1 \end{bmatrix} \underset{\text{by (ii)}}{=} -\delta\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{bmatrix} \underset{\substack{\text{by}\\\text{(iii)}}}{=} -1$$

$$\delta\begin{bmatrix} 1 & & & & \\ & 1 & & 0 & \\ & & \boxed{c} & & \\ 0 & & & 1 & \\ & & & & 1 \end{bmatrix} \underset{\text{by (i)}}{=} c\,\delta\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \underset{\substack{\text{by}\\\text{(iii)}}}{=} c$$

Also, given $A = \begin{bmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{bmatrix} \in \mathbb{R}^{n\times n}$,

$$\delta\left(A\begin{bmatrix} 1 & & \boxed{a} \\ & \ddots & \\ 0 & & 1 \end{bmatrix}\right) = \delta(A_1, \ldots, A_i + aA_j, \ldots, A_j, \ldots, A_n) \underset{\substack{\text{by}\\\text{(i)}}}{=} \delta(A_1, \ldots, A_n) + a\,\underbrace{\delta(A_1, \ldots, A_j, \ldots, A_j, \ldots, A_n)}_{=0 \text{ by (ii)}}$$

$$= \delta(A)$$

$$\left(= \delta(A)\cdot\delta\begin{bmatrix} 1 & \boxed{a} \\ & 1 \end{bmatrix}\right)$$

$$\delta\left(A\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \boxed{\begin{matrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{matrix}} & \\ 0 & & & 1 \end{bmatrix}\right) = \delta(A_1, \ldots, A_j, \ldots, A_i, \ldots, A_n) \underset{\text{by (ii)}}{=} -\delta(A_1, \ldots, A_n) \left(= \delta(A)\cdot\delta\begin{bmatrix} 1 & & \\ & 1 & \\ & & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} \\ & & & 1 \end{bmatrix}\right)$$

$$\delta\left(A\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \boxed{c} & \\ 0 & & 1 & \\ & & & 1 \end{bmatrix}\right) = \delta(A_1, \ldots, cA_i, \ldots, A_n) \underset{\text{by (i)}}{=} c\,\delta(A_1, \ldots, A_n) \left(= \delta(A)\cdot\delta\begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \boxed{c} & \\ 0 & & 1 & \\ & & & 1 \end{bmatrix}\right)$$

Hence if A has column-echelon form $A' = A E_1 E_2 \cdots E_r$,

this shows $\delta(A) = \delta(A') \cdot \delta(E_1) \cdots \delta(E_r)$ where $\delta(E_i) = \begin{cases} 1 & \text{if } E_i \text{ is type (i)} \\ -1 & \text{if } E_i \text{ is type (ii)} \\ c & \text{if } E_i = \begin{bmatrix} 1 & & \\ & \boxed{c} & \\ & & 1 \end{bmatrix} \text{ of type (iii)} \end{cases}$

($=\det E_i$ in each case)

But also, either $A' = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$ so $\delta(A') \underset{\text{by (iii)}}{=} 1$

or $A' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ * & * & * & 0 & 0 \end{bmatrix}$ has some zero column,

so $\delta(A') \underset{\text{by (i)}}{=} 0 \cdot \delta(A') = 0.$

Thus $\delta(A)$ is already uniquely determined by (i), (ii), (iii) ∎

9/19/18 REMARK: This also showed row-reduction lets one calculate $\det A$ in $\leq C \cdot n^3$ steps

The argument in the previous proof really showed more...

THEOREM: For $A \in \mathbb{R}^{n \times n}$, (a) $\det A \neq 0 \iff A$ is invertible

and (b) $\det(AB) = \det A \cdot \det B$.

proof: (a): As in previous proof, if A has col-echelon form A'

then $A' = A E_1 E_2 \cdots E_r$

with $\underbrace{\det(A')}_{\begin{cases} 0 & \text{if } A' \text{ has a zero column, that is, if } A \text{ is not invertible} \\ 1 & \text{if } A' = I_n \text{, that is, if } A \text{ is invertible} \end{cases}} = \det(A) \cdot \underbrace{\det(E_1) \cdots \det(E_r)}_{\text{these are all } +1, -1, \text{ or } c \neq 0}$

Thus $\det(A) = \begin{cases} 0 & \text{if } A \text{ is not invertible} \\ \text{nonzero} & \text{if } A \text{ is invertible (in fact, } \det(A) = \frac{1}{\det(E_1) \cdots \det(E_r)} \text{ in this case)} \end{cases}$

(b): If B is not invertible, then $\det(B) = 0$ by (a), but also AB is not invertible

since any $X \neq 0$ with $BX = 0$ has $ABX = 0$ also. So $\det(AB) = 0$ by (a),

and $\underbrace{\det(AB)}_{0} = \det(A) \cdot \underbrace{\det(B)}_{0}$ holds for trivial reasons.