(19) Hence if A has column-echelon form $A' = A E_1 E_2 \cdots E_r$, this shows $\delta(A) = \delta(A') \cdot \delta(E_1) \cdots \delta(E_r)$ where $\delta(E_i) = \begin{cases} 1 & \text{if } E_i \text{ is type (i)} \\ -1 & \text{if } E_i \text{ is type (ii)} \\ c & \text{if } E_i = \begin{bmatrix} 1 & & \\ & \boxed{c} & \\ & & 1 \end{bmatrix} \text{ of type (iii)} \end{cases}$

(= det E_i in each case)

But also, either $A' = \begin{bmatrix} \textcircled{1} & & 0 \\ & \textcircled{1} & \\ 0 & & \ddots \ \textcircled{1} \end{bmatrix}$ so $\underset{\text{by (iii)}}{\delta(A') = 1}$

or $A' = \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 \\ * & * & * & 0 & 0 \end{bmatrix}$ has some zero column,

so $\delta(A') \underset{\text{by (i)}}{=} 0 \cdot \delta(A') = 0.$

9/19/18 → Thus $\delta(A)$ is already uniquely determined by (i), (ii), (iii) ■

REMARK: This also showed row-reduction lets one calculate det A in $\le C \cdot n^3$ steps

The argument in the previous proof really showed more...

THEOREM: For $A \in \mathbb{R}^{n \times n}$, (a) $\det A \neq 0 \iff A$ is invertible

and (b) $\det(AB) = \det A \cdot \det B$.

proof: (a): As in previous proof, if A has col-echelon form A'

then $A' = A E_1 E_2 \cdots E_r$

with $\underbrace{\det(A')}_{\substack{\| \\ \begin{cases} 0 \text{ if } A' \text{ has a zero column, that is, if } A \text{ is not invertible} \\ 1 \text{ if } A' = I_n \text{, that is, if } A \text{ is invertible} \end{cases}}} = \det(A) \cdot \underbrace{\det(E_1) \cdots \det(E_r)}_{\text{these are all } +1, -1, \text{ or } c \neq 0}$

Thus $\det(A) = \begin{cases} 0 & \text{if } A \text{ is not invertible} \\ \text{nonzero} & \text{if } A \text{ is invertible} \end{cases}$ (in fact, $\det(A) = \dfrac{1}{\det(E_1) \cdots \det(E_r)}$ in this case)

(b): If B is not invertible, then $\det(B) = 0$ by (a), but also AB is not invertible since any $X \neq 0$ with $BX = 0$ has $ABX = 0$ also. So $\det(AB) = 0$ by (a),

and $\underbrace{\det(AB)}_{0} = \det(A) \cdot \underbrace{\det(B)}_{0}$ holds for trivial reasons.

(20)

If B is <u>invertible</u>, we col ops to write $B = E_1 E_2 \cdots E_r$ for elementary matrices E_i, and we've already seen $\det B = \det E_1 \cdots \det E_r$ in this situation. But then also we saw

$$\det(A \underbrace{E_1 E_2 \cdots E_r}_{B}) = \det(A) \cdot \underbrace{\det(E_1) \cdots \det(E_r)}_{\det B} \quad \text{by properties (i), (ii), (iii)}$$

and hence $\det(AB) = \det(A) \cdot \det(B)$. ■

It's helpful to know some other expressions for $\det A$ ...

PROP: (Laplace Row expansion) For each $i = 1, \ldots, n$

$$\det A = (-1)^{i+1}\left(a_{i1} \det(\hat{A}_{i1}) - a_{i2} \det(\hat{A}_{i2}) + \ldots \pm a_{in} \det(\hat{A}_{in})\right)$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\hat{A}_{ij})$$

where here (temporarily!) $\hat{A}_{ij} := A$ with row i column j removed

---

e.g. $n=3$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \overset{\text{Laplace expand along row 2}}{=} -d \det \begin{bmatrix} b & c \\ h & i \end{bmatrix} + e \det \begin{bmatrix} a & c \\ g & i \end{bmatrix} - f \begin{bmatrix} a & b \\ g & h \end{bmatrix}$$

---

proof: Let $\delta(A) :=$ right-side of the PROPOSITION

and check

- $\delta(A) = \delta(A_1, \ldots, A_n)$ is <u>linear</u> in each column A_k because each of the simpler functions $(-1)^{i+j} a_{ij} \det(A_{ij})$ for $j = 1, \ldots, n$ has this property
- $\delta(A) = \delta(A_1, \ldots, A_n)$ is <u>alternating</u> in the columns A_k because if one swaps columns A_r, A_s then it <u>negates</u> $\det(A_{ij})$ for $j \notin \{r, s\}$ while the two terms $\pm a_{ir} \det(A_{ir})$, $\pm a_{is} \det(A_{is})$ swap ~~positions~~ in the sum and <u>are negated</u>.
- $\delta(I_n) = \delta \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} = 1$ is easy to prove by induction on n.

Hence $\delta(A) = \det(A)$ ■

(21) This leads to a formula for A^{-1}.

THEOREM (1.6.9) If $A \in \mathbb{R}^{n\times n}$ is invertible, then A^{-1} has (i,j)-entry given by $\frac{(-1)^{i+j}}{\det A} \det(\hat{A}_{ji})$

---

e.g. $n=2$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det\begin{bmatrix} a & b \\ c & d\end{bmatrix}} \begin{bmatrix} +d & -b \\ -c & +a \end{bmatrix} = \frac{1}{ad-b}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$$

---

proof: Let $B \in \mathbb{R}^{n\times n}$ have $B_{ij} := \frac{(-1)^{i+j}}{\det A} \det(\hat{A}_{ji})$

and compute
$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$
$$= \sum_{k=1}^{n} a_{ik} \cdot \frac{(-1)^{k+j} \det(\hat{A}_{jk})}{\det A}$$
$$= \frac{1}{\det A} \sum_{k=1}^{n} (-1)^{k+j} a_{ik} \det(\hat{A}_{jk})$$
$$= \frac{1}{\det A} \cdot \begin{cases} \text{Laplace expansion of } \det A \text{ along row } i & \text{if } i=j \\ \text{laplace expansion of } \det \begin{bmatrix} - & A_1 & - \\ & \vdots & \\ - & A_i & - \\ & \vdots & \\ - & A_i & - \\ & \vdots & \\ - & A_n & - \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \\ \\ \leftarrow \\ \\ \\ \end{matrix} \text{ along row } i & \text{if } i\neq j \end{cases}$$

i^{th} row repeated as j^{th} row

$$= \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}.$$

That is, $AB = I_n$ ∎

EXERCISE: Show $\det(A^t) = \det A$ using the def'n of $\det A = \sum_{p\in S_n} \text{sign}(p)\, a_{1p(1)} \cdots a_{n,p(n)}$

(COR. 1.4.15 (b))

(22)

Chapter 2 Groups

§2.2, 2.1 Groups & subgroups (& laws of composition)

DEF'N: A *group* is a set G with a *law of composition* $G \times G \to G$, $(a,b) \mapsto ab$ (just a function!)

satisfying
- *associativity*: $(ab)c = a(bc)$ $\forall a,b,c \in G$
- *existence of (2-sided) identity*: $\exists 1 \in G$ with $1 \cdot a = a \cdot 1 = a$ $\forall a \in G$
- *existence of (2-sided) inverses*: $\forall a \in G$ $\exists b \in G$ with $ab = ba = e$ (and b is called a^{-1})

A subset $H \subseteq G$ is called a *subgroup* of G (written $H \leq G$) if it is also a group using the same composition law $G \times G \to G$, that is, it satisfies
- *closure* (under composition): $a,b \in H \Rightarrow \underline{ab} \in H$ ← this lives in G, a priori, maybe not in H
- *existence of identity*: $1 \in H$
- *closure under inverses*: $a \in H \Rightarrow a^{-1} \in H$

9/21/18 →

EXAMPLES: ① S_n = symmetric group (on n letters) $= \{\text{permutations } \{1,2,\ldots,n\} \xrightarrow{p} \{1,2,\ldots,n\}\}$

with Law of Composition: $S_n \times S_n \longrightarrow S_n$, $(p, q) \longmapsto pq$

Q: Who is 1?

$$1 = e = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} \quad \text{since } ep = pe = p \ \forall p \in S_n$$

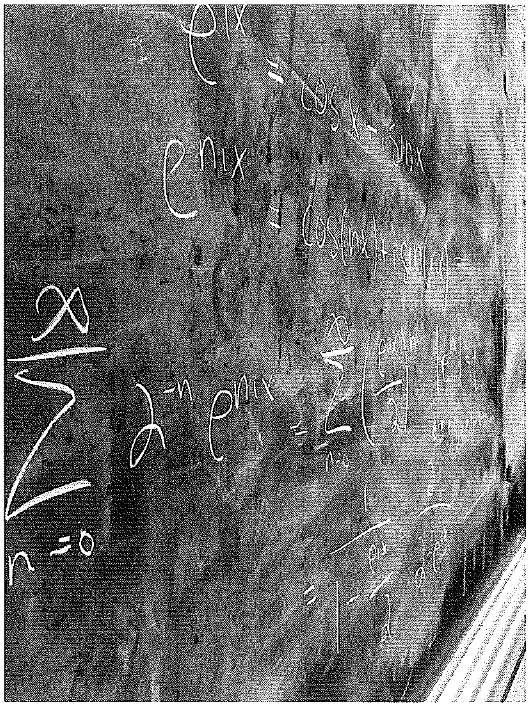
Why associative? $(pq)r = p(qr)$? YES: Both equal the result of doing all 3 in a row here..

$$\begin{tikzcd}
\{1,\ldots,n\} \arrow[r, "r"] \arrow[rr, bend left, "qr"] \arrow[rrr, bend left=40, "p(qr)"] \arrow[rrr, bend right=40, "(pq)r"'] & \{1,\ldots,n\} \arrow[r, "q"] \arrow[rr, bend right, "pq"'] & \{1,\ldots,n\} \arrow[r, "p"] & \{1,\ldots,n\}
\end{tikzcd}$$

Undergraduate Mathematics Research Seminar

University of Minnesota
School of Mathematics

- Come hear about the research that undergraduates do in the mathematics department!
- Present on the mathematics research you do!
- Present on a paper you think would be interesting!



When: **Tuesdays, 12:30PM**
Where: **364 Vincent Hall**
Who: **Any undergraduate student is welcome!**

For details, email chand409@umn.edu.