(3)

Matrix addition, scaling, multiplication satisfy lots of properties that are easy to check, usually following from properties of $+, \times$ in $\mathbb{R}$, e.g.

$$A+B=B+A \qquad c(A+B)=cA+cB \qquad (cA)B=c(AB)=A(cB)$$

$$(A+B)+C=A+(B+C) \qquad A(B+C)=AB+AC$$

$$\boxed{A(BC)=(AB)C} \qquad (A+B)C=AC+BC$$

↖ associativity is not to be taken for granted – it has implications!

Note commutativity fails in general: $AB \neq BA$

sometimes for obvious dimension reasons, e.g.

$$\underset{1\times 2}{A}\,\underset{2\times 1}{B} = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \underset{1\times 1}{[2]} \neq \underset{2\times 2}{\begin{bmatrix}1&1\\1&1\end{bmatrix}} = \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} = \underset{2\times 1}{B}\,\underset{1\times 2}{A}$$

but also even when A, B are square $\mathbb{R}^{n\times n}$

e.g.

$$\underset{A}{\begin{bmatrix}1&1\\0&1\end{bmatrix}}\underset{B}{\begin{bmatrix}0&1\\1&0\end{bmatrix}} = \begin{bmatrix}1&1\\1&0\end{bmatrix} \neq \begin{bmatrix}0&1\\1&1\end{bmatrix} = \underset{B}{\begin{bmatrix}0&1\\1&0\end{bmatrix}}\underset{A}{\begin{bmatrix}1&1\\0&1\end{bmatrix}}$$

9/7/18

The addition of matrices has $0_{m\times n} := \begin{bmatrix}0&0&\cdots&0\\0&0&&\\\vdots&&\ddots&\\0&&&0\end{bmatrix}$ as (additive) identity $0+A=A=A+0$ $\forall A \in \mathbb{R}^{m\times n}$

and $-A=(-1)A$ is the (additive) inverse of A

For multiplication, the $(n\times n)$ identity matrix $I_n = \begin{bmatrix}1&0&\cdots&0\\0&1&0\cdots&0\\\vdots&0&\ddots&\\0&0&&1\end{bmatrix}$ has $I_nA=A$, $AI_m=A$ $\forall A \in \mathbb{R}^{n\times m}$

and [redacted] inverses are more subtle:

Call $L \in \mathbb{R}^{m\times n}$ with $LA = I_m$ a left-inverse for A

$R \in \mathbb{R}^{m\times n}$ with $AR = I_n$ a right-inverse for A

- A might have neither, e.g. $A=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ or $A=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$ or $A=\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}$
- A might have one but not the other, e.g. $A=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$ has any $R=\begin{bmatrix}1&0\\0&1\\r_1&r_2\end{bmatrix}$ as a right-inverse $AR=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I_2$ but check it has no left-inverse.

(3½)

Why is $(AB)C = A(BC)$

for matrices $\underset{m\times n}{A}$, $\underset{n\times p}{B}$, $\underset{p\times q}{C}$?

proof: Compare

$$\Big(\overbrace{(AB)}^{m\times p}\overbrace{C}^{p\times q}\Big)_{i,j} = \sum_{k=1}^{p} (AB)_{ik} C_{kj}$$

$$= \sum_{k=1}^{p} \left(\sum_{l=1}^{n} A_{il} B_{lk}\right) C_{kj}$$

$$= \sum_{k=1}^{p} \sum_{l=1}^{n} A_{il} B_{lk} C_{kj}$$

versus

$$\Big(\overbrace{A}^{m\times n}\overbrace{(BC)}^{n\times q}\Big)_{i,j} = \sum_{l=1}^{n} A_{il} (BC)_{lj}$$

$$= \sum_{l=1}^{n} A_{il} \left(\sum_{k=1}^{p} B_{lk} C_{kj}\right)$$

$$= \sum_{k=1}^{p} \sum_{l=1}^{n} A_{il} B_{lk} C_{kj}$$

SAME!

∎

(4)

Associativity forces this...

<u>PROP</u>: If $A \in \mathbb{R}^{m \times n}$ has both a left- and a right-inverse, then they're the same: $L = R$.
$LA = I_m$ $\quad AR = I_n$

(<u>Q</u>: What dimensions must L, R have?)

<u>proof</u>: Consider $\underbrace{(LA)}_{I_m} R \overset{\uparrow \text{associativity}}{=} L \underbrace{(AR)}_{I_n}$

$R = \quad = L$ ∎

In this case, one says A has a (<u>two-sided</u>) <u>inverse</u>, or just an <u>inverse</u>, and one calls this $L = R =: A^{-1}$, satisfying $A^{-1} \cdot A = I_m$, $A \cdot A^{-1} = I_n$

But what's not yet clear is that this also <u>forces</u> $m = n$ i.e. $A \in \mathbb{R}^{n \times n}$ must be <u>square</u>!

For this, it helps to develop <u>row-reduction</u>.

Recall that when one wants to solve a <u>linear system</u> of m equations in n unknowns

e.g. $m=3$, $n=4$

$$\begin{cases} -2x_1 - 6x_2 - 5x_3 - 11x_4 = 1 \\ x_1 + 3x_2 + 4x_3 + 4x_4 = 1 \\ x_1 + 3x_2 + 6x_3 + 2x_4 = 3 \end{cases}$$

it's helpful to recast it as $AX = Y$ where $A \in \mathbb{R}^{m \times n}$, $X \in \mathbb{R}^n = \mathbb{R}^{n \times 1}$, $Y \in \mathbb{R}^m = \mathbb{R}^{m \times 1}$ (column vectors)

e.g.

$$\underbrace{\begin{bmatrix} -2 & -6 & -5 & -11 \\ 1 & +3 & 4 & 4 \\ 1 & 3 & 6 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}_{Y}$$

and then there are 3 kinds of (invertible) operations one can do to the system, that don't change the solution set to $AX = Y$, and which correspond to 3 types of <u>row operations</u> on the <u>augmented matrix</u> $[\,A \mid Y\,]$ ...
(last column)

Type (i): adding c times i^{th} equation to j^{th} equation, corresponds to adding c times row i to row j in $[A|Y]$

Type (ii): ~~swap~~ re-ordering the equations by swapping i^{th} and j^{th}, corresponds to swapping rows i & j of $[A|Y]$

Type (iii): scaling the i^{th} equation by $c \in \mathbb{R} - \{0\}$, corresponds to scaling i^{th} row of $[A|Y]$

One can always do this until $[A|Y]$ is in row echelon form:

$$\left[\begin{array}{cccccccccc|c} 0\cdots 0 & 1 & *\cdots * & 0 & *\cdots * & 0 & *\cdots * \\ 0\cdots 0 & 0 & 0\cdots 0 & 1 & *\cdots * & 0 & *\cdots * \\ 0\cdots 0 & 0 & 0\cdots 0 & 0 & 0\cdots 0 & 1 & *\cdots * \\ 0 & 0 & \cdots & & & & 0 \\ 0 & 0 & \cdots & & & & 0 \end{array}\right]$$

e.g. $[A|Y] = \left[\begin{array}{cccc|c} -2 & -6 & -5 & -11 & +1 \\ +1 & +3 & +4 & +4 & +1 \\ +1 & +3 & +6 & +2 & +3 \end{array}\right]$

$\downarrow$ add 2 (row 3) ~~to~~ row 1 (Type (i))

$$\left[\begin{array}{cccc|c} 0 & 0 & +7 & -7 & +7 \\ +1 & +3 & +4 & +4 & +1 \\ +1 & +3 & +6 & +2 & +3 \end{array}\right]$$

$\downarrow$ add (-1)(row 2) to row 3 (Type (i))

$$\left[\begin{array}{cccc|c} 0 & 0 & 7 & -7 & 7 \\ 1 & 3 & 4 & 4 & 1 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$\downarrow$ swap rows 1 and 2 (Type (ii))

$$\left[\begin{array}{cccc|c} 1 & 3 & 4 & 4 & 1 \\ 0 & 0 & 7 & -7 & 7 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$\downarrow$ scale row 2 by $\frac{1}{7}$ (Type (iii))

$$\left[\begin{array}{cccc|c} 1 & 3 & 4 & 4 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -2 & 2 \end{array}\right]$$

$\downarrow$ add (-4)(row 2) to row 1
add (-2)(row 2) to row 3

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 8 & -3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longleftarrow \text{row-echelon form} \left[\begin{array}{cccc|c} 1 & * & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which corresponds to the easier system $[A'|Y']$ i.e. $A'X = Y'$ $\left\{\begin{array}{l} x_1 + 3x_2 \quad + 8x_4 = -3 \\ \qquad x_3 - x_4 = 1 \\ \qquad\qquad 0 = 0 \end{array}\right\}$

whose solutions have x_2, x_4 arbitrary in $\mathbb{R}$ and $x_3 = 1 + x_4$
$x_1 = -3 - 3x_2 - 8x_4$

(6)

More formally ...

DEF'N: $M \in \mathbb{R}^{m \times n}$ is in <u>row-echelon</u> <u>form</u> if

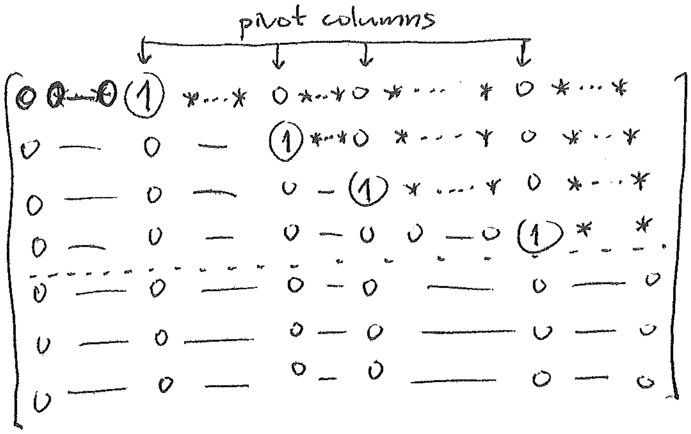
(a) its zero rows all come at the end (row i zero $\Rightarrow$ row j is zero $\forall j \geq i$)

(b) each nonzero row has its leftmost nonzero entry being 1, (called a <u>pivot</u> 1)

(c) if rows i and $i+1$ are nonzero, the pivot 1 in row i is <u>left</u> of the pivot 1 in row $i+1$

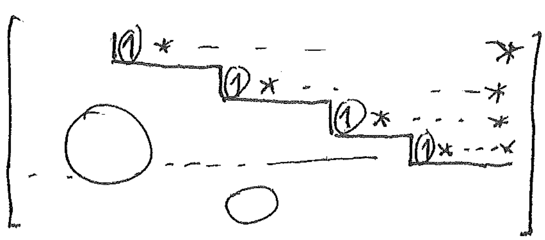
(d) pivot 1's are the only nonzero entry in their column.

e.g.



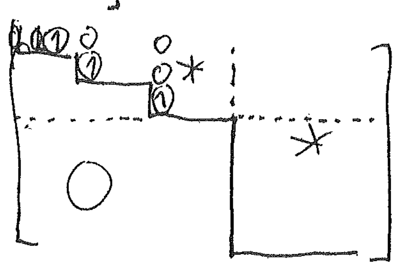
<u>PROPOSITION</u>: Every matrix $M \in \mathbb{R}^{m \times n}$ can be brought to row-echelon form by a sequence of row ops of type (i), (ii), (iii)

<u>proof</u>: Induct on the number of nonzero rows.
(sketch)



(<u>Q</u>: What does the <u>base case</u> of the induction look like?)

In the inductive step, one has it in this form



and one finds the <u>leftmost</u> nonzero entry below the dotted line, scales its row to make it a pivot 1, swaps rows until it is just below the dotted line, then uses the pivot 1 to eliminate entries in the same column. ∎

9/10/18

REMARK: The row-echelon form for M is <u>unique</u>, but we won't need this.