Math 5285 Honors abstract algebra
Spring 2008, Vic Reiner
Final exam - Due Friday May 9, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are not allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (30 points total; 5 points each part)

(a) (5 points) Show that in the symmetric group \( S_n \), conjugating an \( m \)-cycle gives an \( m \)-cycle, and more specifically
\[
\sigma(a_1 a_2 \cdots a_{m-1} a_m)\sigma^{-1} = (\sigma(a_1) \sigma(a_2) \cdots \sigma(a_{m-1}) \sigma(a_m)).
\]

(b) (5 points) Show that in the symmetric group \( S_5 \), the subgroup \( \langle \tau, \sigma \rangle \) generated by any 2-cycle \( \tau = (ij) \) together with any 5-cycle \( (abcde) \) is the whole group \( S_5 \).

(Recall that we wanted this fact from (b) in lecture, in order to conclude that an irreducible quintic polynomial \( f(x) \in \mathbb{Q}[x] \) that had exactly 3 real roots in \( \mathbb{C} \) has Galois group \( G(\text{Split}_\mathbb{Q}(f)/\mathbb{Q}) \) isomorphic to \( S_5 \).

(c) (5 points) Which cycle types (= lists of cycle sizes) for permutations of \( S_5 \) are the ones that lie in the subgroup of alternating permutations \( A_5 \)?

(d) (5 points) Show that the two 5-cycles \((12345)\) and \((21345)\) are conjugate within \( S_5 \), but not conjugate within \( A_5 \).

(e) (5 points) Write down the class equation for \( A_5 \), that is, the list of sizes of all of the conjugacy classes, and how they add up to \(|A_5|\).

(f) (5 points) Prove that a normal subgroup \( H \) of a finite group \( G \) must have its cardinality \(|H|\) expressible as a sum of cardinalities of distinct conjugacy classes in \( G \), and one of these cardinalities must be 1, corresponding to the identity conjugacy class \( \{e\} \).

Use this to deduce that \( A_5 \) is a simple group (i.e. it has no non-identity proper normal subgroups), and hence is not a solvable group. Explain why this proves \( S_5 \) is also not a solvable group.

(Recall that we wanted this to conclude that the quintic polynomials \( f \in \mathbb{Q}[x] \) with exactly 3 real roots mentioned above are not solvable by radicals).
2. (15 points total) Let \( \mathbb{Q} \subset \mathbb{F} \subset \mathbb{K} \) be a Galois extension in characteristic zero, with Galois group \( G(\mathbb{K}/\mathbb{F}) \cong D_4 \), the dihedral group of order 8, the symmetries of a square.

(a) (10 points) How many intermediate subfields \( \mathbb{L} \) are there lying strictly between \( \mathbb{F} \) and \( \mathbb{K} \), that is, with \( \mathbb{F} \subset \mathbb{L} \subset \mathbb{K} \)?

(b) (5 points) How many of the intermediate subfields \( \mathbb{L} \) from part (a) have \( \mathbb{L}/\mathbb{F} \) Galois?

3. (20 points total) Consider the matrix \( A \in \mathbb{Z}^{4 \times 4} \) shown below

\[
A = \begin{bmatrix}
3 & -1 & -1 & -1 \\
-1 & 3 & -1 & -1 \\
-1 & -1 & 3 & -1 \\
-1 & -1 & -1 & 3 \\
\end{bmatrix}
\]

as representing a \( \mathbb{Z} \)-module homomorphism \( \mathbb{Z}^4 \rightarrow \mathbb{Z}^4 \) with respect to the standard basis for \( \mathbb{Z}^4 \) in both the domain and range.

Write the two finitely generated \( \mathbb{Z} \)-modules \( \ker A \) and \( \mathbb{Z}^4 / \text{im} A \) explicitly in the form

\[
\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}/n_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_r \mathbb{Z}
\]
guaranteed by the theorem on finitely generated modules over a Euclidean domain.

4. (15 points total) Artin’s Problem 12.6.4 on page 487.

5. (10 points total) Artin’s Problem 12.7.21 on page 489.

6. (10 points total) Consider the matrix \( A \in \mathbb{C}^{5 \times 5} \) shown below

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

making \( V = \mathbb{C}^5 \) a finitely-generated \( \mathbb{C}[t] \)-module, in which \( t \) acts on an element \( v \) in \( V = \mathbb{C}^5 \) as left-multiplication by \( A \).

Write \( V \) explicitly in the form

\[
\mathbb{C}[t] \oplus \cdots \oplus \mathbb{C}[t] \oplus \mathbb{C}[t]/(f_1(t)) \oplus \cdots \oplus \mathbb{C}[t]/(f_r(t))
\]
guaranteed by the theorem on finitely generated modules over a Euclidean domain.