1. (30 points total; 10 points each part) Define three rings by adjoining roots of quartic polynomials to $F_2 = \mathbb{Z}/2\mathbb{Z}$:

- $R_1 = F_2[x]/(x^4 + x + 1)$
- $R_2 = F_2[x]/(x^4 + x^2 + 1)$
- $R_3 = F_2[x]/(x^4 + x^3 + 1)$

(a) Prove that $R_1, R_3$ are fields, but that $R_2$ is not a field.
(b) Find a primitive element in $R_1$, that is, an element $\beta$ such that $R_1^\times = R_1 - \{0\} = \{1, \beta, \beta^2, \ldots\}$.
(c) Give an explicit ring isomorphism $R_1 \xrightarrow{\varphi} R_3$ (and prove that it is an isomorphism).

2. (15 points) Artin’s Chapter 11 Section 2 Problem 2 part (b) on p. 442.

3. (20 points) Let $f(x) = ax^2 + bx + c$ in $\mathbb{R}[x]$ be any irreducible quadratic polynomial, that is, one with $b^2 - 4ac < 0$. Prove there is a ring isomorphism $\mathbb{R}[x]/(f(x)) \cong \mathbb{C}$.

4. (15 points) A field $F$ is algebraically closed if every nonconstant polynomial $f(x)$ in $F[x]$ has at least one root in $F$. Prove that finite fields $F$ are never algebraically closed.
(Hint: Given a finite field $F$, can you write down an explicit nonconstant polynomial $f(x)$ that has no roots in $F$?)

5. (20 points total; 10 points each) Two problems from Artin on rings of formal power series:
(a) Artin’s Chapter 10 Section 2 Problem 6 part (b) on p. 380.
(b) Artin’s Chapter 10 Section 3 Problem 26 on p. 382.