Math 5285 Honors abstract algebra  
Spring 2008, Vic Reiner  
Midterm exam 2- Due Wednesday April 16, in class  
Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are not allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points) Let $\alpha$ be the positive real 4th root of 2 and let $\beta$ be the positive real square root of 3. Find an element $\gamma$ in $K = \mathbb{Q}(\alpha, \beta)$ for which $K = \mathbb{Q}(\gamma)$ (and prove it).

2. (20 points total; 5 points each part) As in Problem 1, let $\alpha$ be the positive real 4th root of 2. Let $K = \mathbb{Q}(\alpha)$.
   (a) Compute the degree $[K : \mathbb{Q}]$ (with proof).
   (b) Prove or disprove: there exists an element $\sigma$ of the Galois group $G(K/\mathbb{Q})$ that negates $\alpha^2$, that is, that sends $\sqrt{2} \to -\sqrt{2}$.
   (c) Describe the Galois group $G(K/F)$.
   (d) Is $K/F$ a Galois extension? Explain your answer.

3. (30 points total; 15 points each part) Let $f(x)$ in $\mathbb{Q}[x]$ have form as in (a) or (b) below. By computing $\gcd(f(x), f'(x))$ in $\mathbb{Q}[x]$, find an integer polynomial expression $D(b, c)$ (that is, an expression $D(b, c)$ lying in $\mathbb{Z}[b, c]$) with the following property: $f(x)$ has multiple roots when one passes to its splitting field if and only if $D(b, c) = 0$.
   (a) $f(x) = x^2 + bx + c$ in $\mathbb{Q}[x]$ is a monic quadratic polynomial.
   (b) $f(x) = x^3 + bx + c$ in $\mathbb{Q}[x]$ is a monic cubic polynomial that has zero coefficient on the $x^2$ term.

4. (30 points total; 10 points each part.) Let $p$ be a prime number. Find, as a function of $p$, the number of irreducible polynomials of degree $d$ in $\mathbb{F}_p[x]$ when
   (a) $d = 2$,
   (b) $d = 3$,
   (c) $d = 4$.
   (Hint for all parts: How does the irreducible factorization of $x^{p^d} - x$ in $\mathbb{F}_p[x]$ look?)

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\[ which can always be obtained from a monic cubic polynomial $x^3 + ax^2 + bx + c$ by replacing $x$ with $x - \frac{a}{3}$. \]