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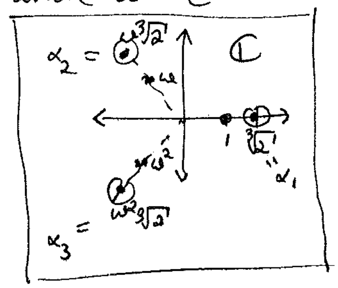
EXAMPLE: Let's use some of this to start analyzing the extension $\mathbb{Q} \subset \mathbb{K} := \mathbb{Q}(\{\text{all the roots of } x^3-2\})$

$$= \mathbb{Q}(\underset{\alpha_1}{\sqrt[3]{2}},\ \underset{\alpha_2}{\omega\sqrt[3]{2}},\ \underset{\alpha_3}{\omega^2\sqrt[3]{2}})$$

where $\omega := e^{2\pi i/3}$



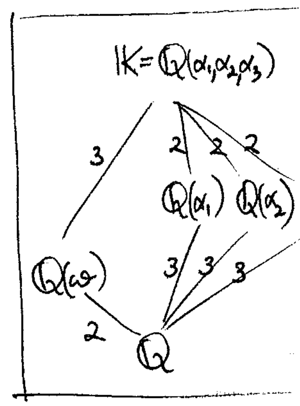
What is $[\mathbb{K}:\mathbb{Q}]$?

Certainly finite, and $\leq 3\cdot 3\cdot 3 = 27$

because of $\underbrace{\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2})}_{\leq 3} \underbrace{\subset \mathbb{Q}(\alpha_1,\alpha_2)}_{\substack{\leq 3 \\ (\text{why?})}} \underbrace{\subset \mathbb{Q}(\alpha_1,\alpha_2,\alpha_3)}_{\substack{\leq 3 \\ (\text{why?})}} = \mathbb{K}$, where $\sqrt[3]{2} = \alpha_1$

But in fact, since $\frac{\alpha_2}{\alpha_1} = \frac{\omega\sqrt[3]{2}}{\sqrt[3]{2}} = \omega = \frac{\alpha_3}{\alpha_2}$ $\left(\text{and } \frac{\alpha_3}{\alpha_1} = \omega^2\right)$

one has $\mathbb{K} = \mathbb{Q}(\alpha_1,\alpha_2,\alpha_3) = \underset{\substack{\text{for any}\\ \text{of } i=1,2,3}}{\mathbb{Q}(\omega,\alpha_i)} = \underset{\substack{\text{for any}\\ i\neq j}}{\mathbb{Q}(\alpha_i,\alpha_j)}$



Note that $m_{\mathbb{Q},\omega}(x) = \underbrace{x^2+x+1}_{\substack{\text{irred. in } \mathbb{Q}[x]\\ \text{via rational root test}}}$ since $\omega^2+\omega+1=0$

so $\underbrace{\mathbb{Q} \subset \mathbb{Q}(\omega)}_{2} \underbrace{\subset \mathbb{Q}(\omega,\alpha_1)}_{\substack{\leq 3\\ (\text{why?})}} = \mathbb{K}$ $\Rightarrow$ $\boxed{[\mathbb{K}:\mathbb{Q}] \leq 2\cdot 3 = 6}$

However $[\mathbb{K}:\mathbb{Q}] = [\mathbb{K}:\mathbb{Q}(\omega)]\underbrace{[\mathbb{Q}(\omega):\mathbb{Q}]}_{2}$ is divisible by 2

and $[\mathbb{K}:\mathbb{Q}] = [\mathbb{K}:\mathbb{Q}(\sqrt[3]{2})]\underbrace{[\mathbb{Q}(\sqrt[3]{2}):\mathbb{Q}]}_{3}$ is divisible by 3

so $\boxed{[\mathbb{K}:\mathbb{Q}] \text{ is divisible by } \operatorname{lcm}(2,3)=6}$

$\Rightarrow [\mathbb{K}:\mathbb{Q}] = 6$

Note that this also implies $[\mathbb{Q}(\omega,\alpha_i):\mathbb{Q}(\omega)] = \frac{6}{2} = 3$, so x^3-2 is still irreducible in $\mathbb{Q}(\omega)[x]$

and $[\mathbb{Q}(\omega,\alpha_i):\mathbb{Q}(\alpha_i)] = \frac{6}{3} = 2$, so x^2+x+1 is still irreducible in $\mathbb{Q}(\alpha_i)[x]$

COROLLARY (15.3.8) Given two extensions $\mathbb{F} \begin{matrix} \subset \mathbb{K}_1 \subset \\ \subset \mathbb{K}_2 \subset \end{matrix} \mathbb{K}$

that are finite, $[\mathbb{K}_1 : \mathbb{F}] = n_1 < \infty$, $[\mathbb{K}_2 : \mathbb{F}] = n_2 < \infty$, then the compositum

$\mathbb{K}_1\mathbb{K}_2 :=$ smallest subfield of $\mathbb{K}$ containing $\mathbb{K}_1, \mathbb{K}_2$

$\left(= \bigcap\limits_{\substack{\text{subfields } \mathbb{K}' \subset \mathbb{K}: \\ \mathbb{K}_1, \mathbb{K}_2 \subset \mathbb{K}'}} \mathbb{K}' = \left\{ \sum_{i=1}^{t} \alpha_i \beta_i : \alpha_i \in \mathbb{K}_1, \beta_i \in \mathbb{K}_2 \right\} \right)$

has $[\mathbb{K}_1\mathbb{K}_2 : \mathbb{F}] \leq n_1 n_2$ (and finite, in particular)

with n_1, n_2 dividing $[\mathbb{K}_1\mathbb{K}_2 : \mathbb{F}]$.

proof: We have

$$\begin{tikzcd}[column sep=small]
 & \mathbb{K} \arrow[d, dash] & \\
 & \mathbb{K}_1\mathbb{K}_2 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{K}_1 \arrow[dr, dash] & & \mathbb{K}_2 = \mathbb{F}(\alpha_1, \ldots, \alpha_m) \text{ for some } \alpha_i \\
 & \mathbb{F} \arrow[ur, dash] &
\end{tikzcd}$$

since $[\mathbb{K}_2 : \mathbb{F}]$ is finite

so consider

$$\begin{tikzcd}[column sep=small]
 & & & \mathbb{K} \arrow[d, dash] & \\
 & & & \mathbb{K}_1\mathbb{K}_2 = \mathbb{K}_1(\alpha_1, \ldots, \alpha_m) \arrow[dr, dash] & \\
 & & \mathbb{K}_1(\alpha_1, \alpha_2) \arrow[ur, dotted, dash] \arrow[dr, dash] & & \mathbb{K}_2 = \mathbb{F}(\alpha_1, \ldots, \alpha_m) \\
 & \mathbb{K}_1(\alpha_1) \arrow[ur, dash] \arrow[dr, dash] & & \mathbb{F}(\alpha_1, \alpha_2) \arrow[ur, dotted, dash] & \\
\mathbb{K}_1 \arrow[ur, dash] \arrow[dr, dash] & & \mathbb{F}(\alpha_1) \arrow[ur, dash] & & \\
 & \mathbb{F} \arrow[ur, dash] & & &
\end{tikzcd}$$

So it's enough to show at each stage

$$\begin{tikzcd}[column sep=small]
 & \mathbb{K}_1(\alpha_1, \ldots, \alpha_r, \alpha_{r+1}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{K}_1(\alpha_1, \ldots, \alpha_r) \arrow[dr, dash] & & \mathbb{F}(\alpha_1, \ldots, \alpha_r, \alpha_{r+1}) \\
 & \mathbb{F}(\alpha_1, \ldots, \alpha_r) \arrow[ur, dash] &
\end{tikzcd}$$

$\deg = \deg\left(m_{\mathbb{F}(\alpha_1, \ldots, \alpha_r), \alpha_{r+1}}(x)\right)$

$\deg = \deg\left(m_{\mathbb{K}_1(\alpha_1, \ldots, \alpha_r), \alpha_{r+1}}(x)\right)$

$\leq \deg\left(m_{\mathbb{F}(\alpha_1, \ldots, \alpha_r), \alpha_{r+1}}(x)\right)$ (why?) ∎

EXAMPLE:

$$\begin{tikzcd}[column sep=small]
 & \mathbb{K} = \mathbb{Q}(\omega\sqrt[3]{2}, \sqrt[3]{2}) = \mathbb{Q}(\omega, \sqrt[3]{2}) \arrow[dl, dash, "2"'] \arrow[dr, dash, "2"] \arrow[dd, dash, "6"] & \\
\mathbb{Q}(\omega\sqrt[3]{2}) \arrow[dr, dash, "3"'] & & \mathbb{Q}(\sqrt[3]{2}) \\
 & \mathbb{Q} \arrow[ur, dash, "3"'] &
\end{tikzcd}$$

$[\mathbb{K} : \mathbb{Q}] = 6 \leq 3 \cdot 3 = 9$

Note $[\mathbb{Q}(\omega\sqrt[3]{2}, \sqrt[3]{2}) : \mathbb{Q}(\sqrt[3]{2})] = 2$ so $\deg(\overbrace{m_{\mathbb{Q}(\sqrt[3]{2}), \omega\sqrt[3]{2}}(x)}^{f(x)}) = 2$, not 3.

But what *is* $f(x)$? $x^3 - 2 = (x - \sqrt[3]{2})(x - \omega\sqrt[3]{2})(x - \omega^2\sqrt[3]{2})$

$= (x - \sqrt[3]{2})\underbrace{(x^2 + \sqrt[3]{2}x + (\sqrt[3]{2})^2)}_{\text{this is } f(x) \,\in\, \mathbb{Q}(\sqrt[3]{2})[x]}$