(66)

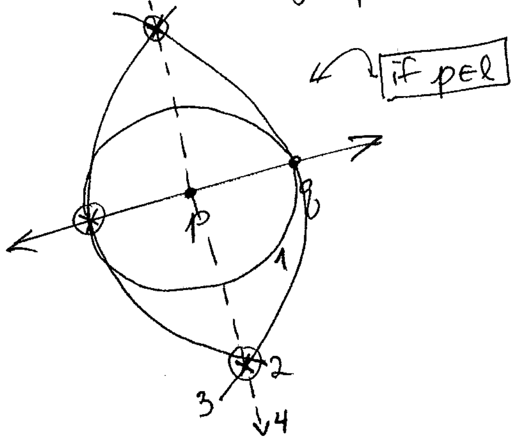
3/15/2019 **PROPOSITION:** $\mathbb{K}_{con}$ is a subfield of $\mathbb{R}$, closed under square roots, and hence $\mathbb{K}_{con} \supseteq \{\alpha \in \mathbb{R} : \exists$ a tower of extensions with

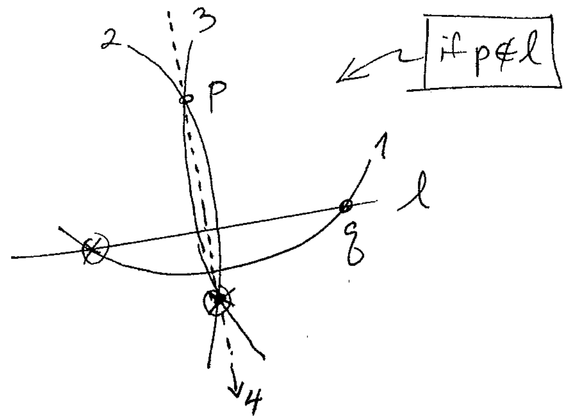
$$\underset{\mathbb{K}_0}{\underset{\parallel}{\mathbb{Q}}} \subset \underset{\mathbb{K}_0(\sqrt{\alpha_1})}{\underset{\parallel}{\mathbb{K}_1}} \subset \underset{\mathbb{K}_1(\sqrt{\alpha_2})}{\underset{\parallel}{\mathbb{K}_2}} \subset \cdots \subset \underset{\mathbb{K}_{r-1}(\sqrt{\alpha_r})}{\underset{\parallel}{\mathbb{K}_r}} \ni \alpha\}$$

$$\boxed{\text{e.g. } \mathbb{K}_{con} \ni \sqrt[4]{\sqrt{\tfrac{2}{3}+\sqrt{17}+\sqrt{\tfrac{9}{23}}} + \frac{4}{\sqrt{6+\sqrt{11}}}} + \frac{1}{\sqrt{35}}}$$

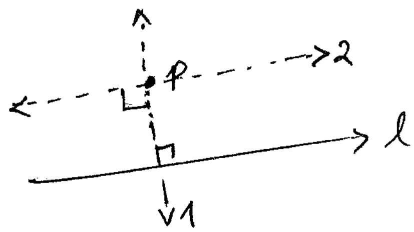
proof: Need to "recall" some constructions...

① Construct a line through $p \perp$ line ℓ:

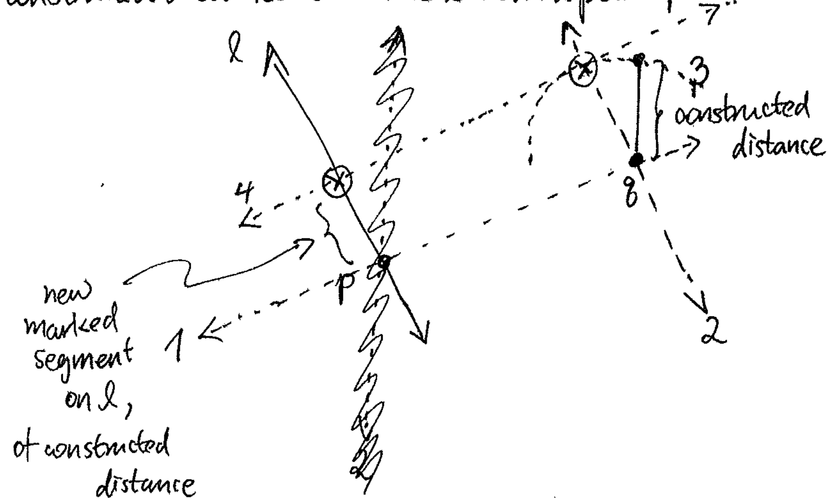




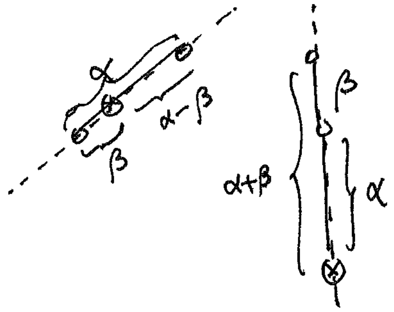
② Pass a parallel to line ℓ through $p \notin \ell$:



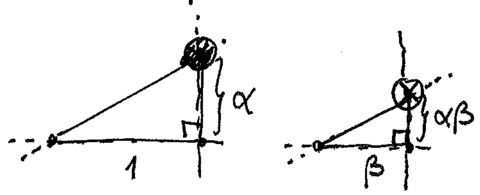
③ Mark a constructed distance onto line ℓ from a point $p \in \ell$:



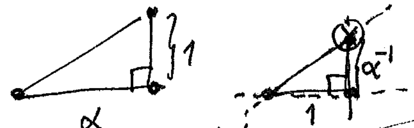
Now given $\alpha, \beta \in \mathbb{K}_{con}$, must show $\alpha \pm \beta$ and $\alpha\beta$, $\alpha^{-1} \in \mathbb{K}_{con}$:



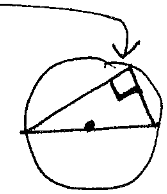
and for $\alpha\beta$ use similar triangles here:



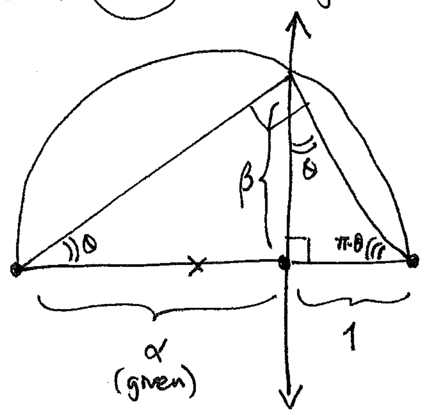
while for α^{-1} use similar triangles here:



The construction of $\sqrt{\alpha}$ uses the fact that this is always a right angle:



Do this construction given α, to get $\sqrt{\alpha}$:



similar triangles gives

$$\frac{\beta}{\alpha} = \frac{1}{\beta} \Rightarrow \beta^2 = \alpha$$

$$\beta = \sqrt{\alpha} \quad \blacksquare$$

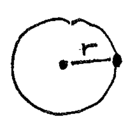
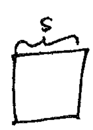
We'll show the following...

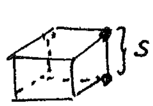
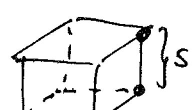
THEOREM: $\mathbb{K}_{con} = \{ \alpha \in \mathbb{R} : \exists \text{ a tower } \underset{\mathbb{K}_0}{\overset{\shortparallel}{\mathbb{Q}}} \subset \underset{\mathbb{K}_0(\sqrt{\alpha_1})}{\overset{\shortparallel}{\mathbb{K}_1}} \subset \underset{\mathbb{K}_1(\sqrt{\alpha_2})}{\overset{\shortparallel}{\mathbb{K}_2}} \subset \cdots \subset \underset{\mathbb{K}_{r-1}(\sqrt{\alpha_r})}{\overset{\shortparallel}{\mathbb{K}_r}} \ni \alpha \}$

and hence every $\alpha \in \mathbb{K}_{con}$ has $[\mathbb{Q}(\alpha):\mathbb{Q}]$ a power of 2 (since it divides $[\mathbb{K}_r:\mathbb{Q}] = 2^r$).

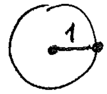
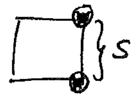
Believing this, one has:

COROLLARY: Using straightedge and compass, one cannot

(i) "square" arbitrary circles: given  find  with same area

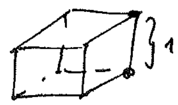
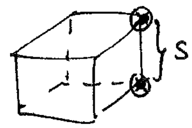
(ii) "duplicate" arbitrary cubes: given  find  with twice the volume

(iii) "trisect arbitrary angles: 

(68) proof: (i) If  and  have same area (constructed), then

$$s^2 = \pi \cdot 1^2 = \pi \quad \Rightarrow \quad s = \sqrt{\pi} \in \mathbb{K}_{con}$$
$$\Rightarrow s^2 = \pi \in \mathbb{K}_{con}$$

But Lindemann showed π is transcendental, not algebraic!

(ii) If  and  have doubled the volume (constructed), then

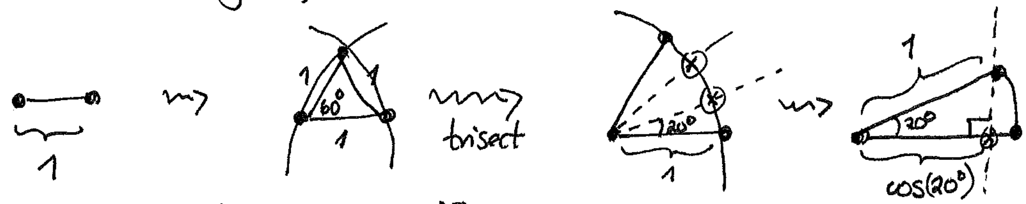
$$s^3 = 2 \cdot 1^3 = 2$$
$$\Rightarrow s = \sqrt[3]{2} \in \mathbb{K}_{con}$$

But $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ since $m_{\mathbb{Q},\sqrt[3]{2}}(x) = x^3 - 2$

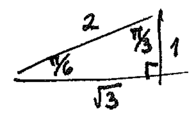
not a power of 2

(iii) If we could trisect angles, then $\alpha = \cos(20^\circ) = \cos\left(\frac{\pi}{9}\right) \in \mathbb{K}_{con}$

as follows:



But then $\beta = 2\alpha = 2\cos\left(\frac{\pi}{9}\right) = e^{i\pi/9} + e^{-i\pi/9} \in \mathbb{K}_{con}$

and $\beta^3 = e^{\frac{i\pi}{3}} + 3e^{\frac{i\pi}{9}} + 3e^{\frac{-i\pi}{9}} + e^{\frac{-i\pi}{3}}$ $\quad \cos\frac{\pi}{3} = \frac{1}{2}$ 

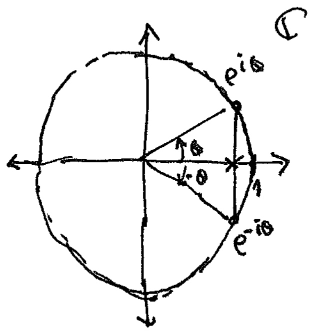
$$= 2 \cdot \frac{1}{2} + 3\beta = 3\beta + 1$$

$\Rightarrow \beta^3 - 3\beta + 1 = 0$, so β is a root of $x^3 - 3x + 1$ irreducible by rational root test (or reduction mod 2)

$\Rightarrow m_{\mathbb{Q},\beta}(x) = x^3 - 3x + 1$

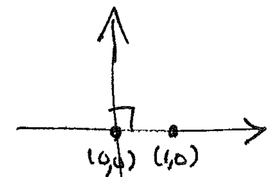
so $[\mathbb{Q}(\beta) : \mathbb{Q}] = 3$, not a power of 2. Contradiction ∎

$2\cos\theta = e^{i\theta} + e^{-i\theta}$



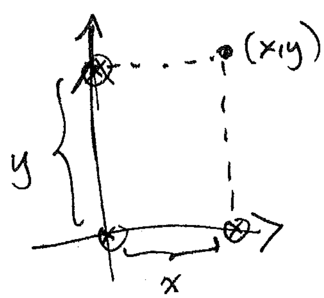
proof of THEOREM: To show every $\alpha \in \mathbb{K}_{con}$ lies in some tower

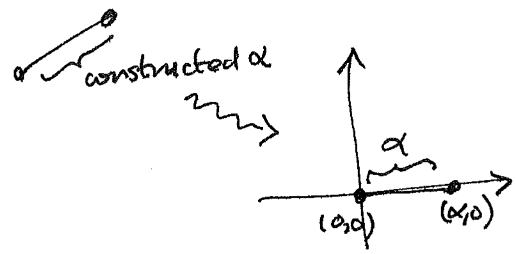
$$\begin{array}{ccccccccc} \mathbb{Q} & \subset & \mathbb{K}_1 & \subset & \mathbb{K}_2 & \subset & \cdots & \subset & \mathbb{K}_r \ni \alpha \\ \| & & \| & & \| & & & & \| \\ \mathbb{K}_0 & & \mathbb{K}_0(\sqrt{\alpha_1}) & & \mathbb{K}_1(\sqrt{\alpha_2}) & & & & \mathbb{K}_{r-1}(\sqrt{\alpha_r}) \end{array}$$



introduce coordinates in $\mathbb{R}^2$ using the original $(0,0),(1,0)$

and note that $\mathbb{K}_{con} = \{$ x or y appearing in any constructed point (x,y) $\}$:

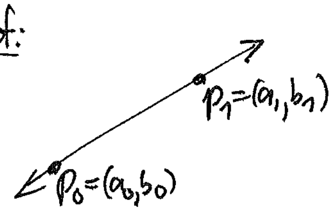




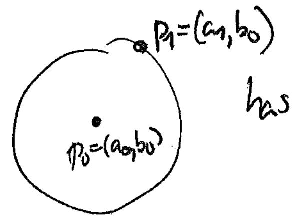
The theorem follows by induction on the number of construction steps used to reach $p=(x,y)$ if we can show these facts:

PROPOSITION (13.5.5): (a) If p_0, p_1 have coordinates in a field $\mathbb{K} \subset \mathbb{R}$, the line $\overleftrightarrow{p_0p_1}$ has equation $ax+by=c$ with $a,b,c \in \mathbb{K}$ and circle centered at p_0 through p_1 has a quadratic equation with coefficients in $\mathbb{K}$. (b) Intersecting two such lines gives a point with coefficients in $\mathbb{K}$, and intersecting such a circle and line or two circles gives points whose coefficients lie in $\mathbb{K}$ or some $\mathbb{K}(\sqrt{\alpha})$ where $\alpha \in \mathbb{K}$

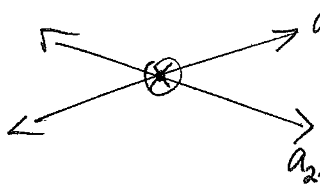
proof:



has equation $\dfrac{y-b_0}{x-a_0} = \dfrac{b_1-b_0}{a_1-a_0}$ i.e. $\underbrace{(a_1-a_0)(y-b_0)=(b_1-b_0)(x-a_0)}_{ax+by=c \text{ with coefficients in } \mathbb{K}}$



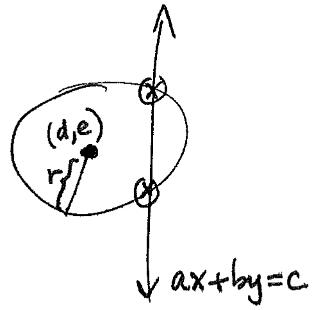
has equation $(x-a_0)^2+(y-b_0)^2=(a_1-a_0)^2+(b_1-b_0)^2$



intersection comes from ~~[illegible]~~ solution to $\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$\Longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

entries in $\mathbb{K}$ if $A \in \mathbb{K}^{2\times 2}$.

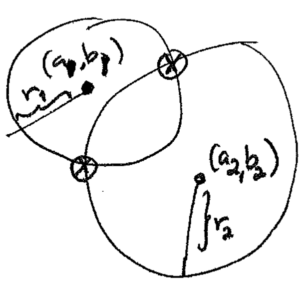


intersections come from solutions to
$$(x-d)^2+(y-e)^2=r^2$$
$$ax+by=c$$

Assuming $b\neq 0$ WLOG, write $y=\frac{c-ax}{b}$

substitute $(x-d)^2+\left(\frac{c-ax}{b}-e\right)^2=r^2$

giving a quadratic equation for x of form
$$Ax^2+Bx+C=0 \quad \text{with } A,B,C\in\mathbb{K}$$
$$\Longrightarrow x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}\in\mathbb{K}\left(\sqrt{B^2-4AC}\right)$$
and then $y=\frac{c-ax}{b}\in\mathbb{K}\left(\sqrt{B^2-4AC}\right)$



intersections come from solutions to
$$\left\{\begin{matrix} (x-a_1)^2+(y-b_1)^2=r_1^2 \\ (x-a_2)^2+(y-b_2)^2=r_2^2 \end{matrix}\right\} \text{ or } \left\{\begin{matrix} x^2-2a_1x+a_1^2+y^2-2b_1y+b_1^2=r_1^2 \\ x^2-2a_2x+a_2^2+y^2-2b_2y+b_2^2=r_2^2 \end{matrix}\right\}$$

subtract

$$\underbrace{2(a_2-a_1)x+a_1^2-a_2^2+2(b_2-b_1)y+b_1^2-b_2^2=r_1^2-r_2^2}_{\text{linear, of form } Ax+By=C \text{ with } A,B,C\in\mathbb{K}}$$

Hence solutions to $(x-a_1)^2+(y-b_1)^2=r_1^2$
$$Ax+By=C$$
again come from a quadratic equation on x, giving $x\in\mathbb{K}(\sqrt{\alpha})$,
$y\in\mathbb{K}(\sqrt{\alpha})$. ∎