3/4/2019

# Chap. 15 Fields and field extensions

We are going to want to understand more about

- ring extensions $\underset{\text{ring}}{R} \overset{\text{subring}}{\subset} \underset{\text{ring}}{S}$
- field extensions $\underset{\text{field}}{F} \overset{\text{subfield}}{\subset} \underset{\text{field}}{K}$

such as $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2},\sqrt{3}) \subset \ldots \subset \mathbb{R} \subset \mathbb{C}$

$\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} : a,b \in \mathbb{Q}\}$

$\mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$

versus $\mathbb{Q} \subset \mathbb{Q}[\pi] \subsetneq \mathbb{Q}(\pi) \subset \mathbb{R}$, where $\pi = 3.14159\ldots$

$\mathbb{Q}[\pi] = \{a_0 + a_1\pi + a_2\pi^2 + \ldots + a_n\pi^n : a_i \in \mathbb{Q}\}$

$\mathbb{Q}(\pi) = \left\{\frac{f(\pi)}{g(\pi)} : f(x), g(x) \in \mathbb{Q}[x], g(\pi) \neq 0\right\}$

## §15.2 Algebraic versus transcendental elements

Let's first clarify the two ways we might add elements...

DEF'N-PROP: For $\underset{\text{ring}}{R} \overset{\text{subring}}{\subset} \underset{\text{ring}}{S}$ and $\alpha \in S$, define

$R[\alpha] :=$ smallest subring of $S$ containing $R, \alpha$

$:=$ subring of $S$ generated by $R, \alpha$

$:= \{ r_0 + r_1\alpha + r_2\alpha^2 + \ldots + r_n\alpha^n : r_i \in R\}$

$= \operatorname{im}\left( R[x] \longrightarrow S \right)$, $x \longmapsto \alpha$, $f(x) \longmapsto f(\alpha)$

$= \bigcap_{\substack{\text{subrings } R' \subset S \\ \text{with } R \subset R'}} R'$

EXERCISE: Check this!

e.g. $\mathbb{Z}[\sqrt{2}] \subset \mathbb{R}$

$\mathbb{Z}[\sqrt{2}] = \{a+b\sqrt{2} : a,b \in \mathbb{Z}\}$

$\mathbb{Z}[\sqrt{2}] \subset \mathbb{Q}[\sqrt{2}] = \{a+b\sqrt{2} : a,b \in \mathbb{Q}\} \subset \mathbb{R}$

**DEF'N-PROP:** Given $\underset{\text{field}}{\mathbb{F}} \overset{\text{subfield}}{\subset} \underset{\text{field}}{\mathbb{K}}$ and $\alpha \in \mathbb{K}$, define

$$\mathbb{F}(\alpha) := \underline{\text{smallest subfield of } \mathbb{K} \text{ containing } \mathbb{F}, \alpha}$$

$$:= \underline{\text{subfield of } \mathbb{K} \text{ generated by } \mathbb{F}, \alpha}$$

$$:= \left\{ \frac{c_0 + c_1\alpha + \cdots + c_n\alpha^n}{d_0 + d_1\alpha + \cdots + d_m\alpha^m} : c_i, d_j \in \mathbb{F}, \ \underset{\sum_i d_i\alpha^i}{\text{denominator}} \neq 0 \right\}$$

EXERCISE: Check this!

$$= \bigcap_{\substack{\text{subfields } F' \subset K \\ \text{with } F \subset F'}} F'$$

$$\cong \text{Frac}(\mathbb{F}[\alpha])$$ via the isomorphism sending $\left( f(\alpha)/g(\alpha) \right) \in \text{Frac}(\mathbb{F}[\alpha])$ to $\frac{f(\alpha)}{g(\alpha)} = g(\alpha)^{-1} f(\alpha)$

EXERCISE: check this!

**EXAMPLES:** ① $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$

$$\mathbb{Q}(\sqrt{2}) = \left\{ \frac{a + b\sqrt{2}}{c + d\sqrt{2}} : a,b,c,d \in \mathbb{Q}, \ c,d \text{ not both } 0 \right\} = \{ a' + b'\sqrt{2} : a', b' \in \mathbb{Q} \} = \mathbb{Q}[\sqrt{2}]$$

$$\frac{a+b\sqrt{2}}{c+d\sqrt{2}} \cdot \frac{c-d\sqrt{2}}{c-d\sqrt{2}} = \frac{ac - 2bd + (bc-ad)\sqrt{2}}{c^2 - 2d^2}$$

② $\mathbb{Q} \subset \mathbb{Q}[\pi] \subsetneq \mathbb{Q}(\pi) \subset \mathbb{R}$

$$\mathbb{Q}[\pi] = \left\{ \sum_{i=0}^{n} a_i \pi^i : a_i \in \mathbb{Q} \right\}, \qquad \mathbb{Q}(\pi) = \left\{ \frac{f(\pi)}{g(\pi)} : f(x), g(x) \in \mathbb{Q}[x], \ g(\pi) \neq 0 \right\}$$

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One also has similar notions of $R[\alpha_1, \ldots, \alpha_n] \subset S$ given $R \overset{\text{subring}}{\subset} S$ and $\alpha_1, \ldots, \alpha_n \in S$

$\mathbb{F}(\alpha_1, \ldots, \alpha_n) \subset \mathbb{K}$ given $\underset{\text{field}}{\mathbb{F}} \overset{\text{subfield}}{\subset} \underset{\text{field}}{\mathbb{K}}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$

e.g. $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \subset \mathbb{R}$

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}[\sqrt{2}, \sqrt{3}] = \{ a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} : a,b,c,d \in \mathbb{Q} \}$$

There are two very different alternatives for $\mathbb{F}(\alpha) \subset \mathbb{K}$

**PROP-DEF'N:** For a field extension $\mathbb{F} \subset \mathbb{K}$ and $\alpha \in \mathbb{K}$, either

(15.2.3
15.2.6
15.2.7)

(a) $\alpha$ is <u>algebraic over $\mathbb{F}$</u>, meaning $\exists\, f(x) \overset{\neq 0}{\in} \mathbb{F}[x]$ with $f(\alpha)=0$,

in which case there is a unique choice of such a monic polynomial $f(x)$ <u>of minimal degree</u> (or monic and <u>irreducible</u>), called the <u>minimal or irreducible polynomial for $\alpha$ over $\mathbb{F}$</u>, (or $m_{\mathbb{F},\alpha}(x)$)

namely the one with $\ker\left( \mathbb{F}[x] \to \mathbb{K} \right) = (f(x))$.
$x \mapsto \alpha$

Furthermore, in this case $\mathbb{F}[\alpha] = \mathbb{F}(\alpha)$ is a field,

with $\mathbb{F}$-basis $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$ where $n = \deg(f)$,

and $f(x)$ is <u>irreducible</u> in $\mathbb{F}[x]$, with $\mathbb{F}[x]/(f(x)) \cong \mathbb{F}[\alpha] = \mathbb{F}(\alpha)$

OR (b) $\alpha$ is <u>transcendental over $\mathbb{F}$</u>, meaning $\nexists\, f(x) \overset{\neq 0}{\in} \mathbb{F}[x]$ with $f(\alpha)=0$.

Furthermore, in this case

$$\begin{array}{ccc} \mathbb{F}[\alpha] & \subsetneq & \mathbb{F}(\alpha) \\ \| \wr & & \| \wr \\ \mathbb{F}[x] & \subsetneq & \mathbb{F}(x) \\ \text{polynomials} & & \text{rational functions} \end{array}$$

**EXAMPLES** ① $\sqrt{2} \in \mathbb{R}$ is algebraic over $\mathbb{Q}$, with minimal polynomial $f(x) = x^2 - 2 \in \mathbb{Q}[x]$

② $\pi \in \mathbb{R}$, $e \in \mathbb{R}$ is transcendental over $\mathbb{Q}$ (but it's not obvious - proven by Hermite for $e$ 1873, by Lindemann for $\pi$ 1882)

③ Consider $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt[4]{2}) \subset \mathbb{R}$ and $\alpha = \sqrt[4]{2} \in \mathbb{R}$

Then $\alpha$ is algebraic over $\mathbb{Q}$, with min poly. (over $\mathbb{Q}$) $m_{\mathbb{Q},\alpha}(x) = x^4 - 2 \in \mathbb{Q}[x]$
(irred. by Eisenstein at $p=2$)

But $\alpha$ is also algebraic over $\mathbb{Q}(\sqrt{2})$,

with min. poly (over $\mathbb{Q}(\sqrt{2})$) $m_{\mathbb{Q}(\sqrt{2}),\alpha}(x) = x^2 - \sqrt{2} \in \mathbb{Q}(\sqrt{2})[x]$.