Math 5705 Undergraduate enumerative combinatorics  
Fall 2002, Vic Reiner  
Midterm exam 3- Due Friday November 15, in class

**Instructions:** This is an open book, open library, open notes, take-home exam, but you are **not** allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (16 points total; 4 points each) Chapter 3, Supplementary problem 2(a),(c),(d),(e) on page 72.

2. (16 points) Chapter 3, Supplementary problem 3 on page 73.

3. (16 points) Chapter 3, Supplementary problem 8 on page 73.

4. (16 points) Chapter 4, Supplementary problem 1 on page 100.

5. (16 points) Chapter 4, Supplementary problem 14 on page 102.

6. This problem is about Stirling numbers of the 1st kind, and is related to #147-149 in the text and also to Chapter 3 Supplementary Problem 10. So feel free to look at those for ideas and hints.

Recall that the **Stirling number of the 2nd kind** $S(k,n)$ is the number of partitions of $[k]$ into $n$ blocks. We showed (Problem # 145) that they are the change-of-basis coefficients in the vector space of polynomials in $x$ of degree at most $k$, writing the basis $\{1, x, x^2, \ldots, x^k\}$ in terms of the basis $\{1, x^1, x^2, \ldots, x^k\}$:

$$ x^k = \sum_{n=0}^{k} S(k,n)x^n, $$

where we further recall that

$$
\begin{align*}
    x^n &:= x(x-1)(x-2)\cdots (x-n+1) \\
    x^n &:= x(x+1)(x+2)\cdots (x+n-1).
\end{align*}
$$

Now define the **Stirling number of the 1st kind** $s(k,n)$, and also the **signless Stirling number of the 1st kind** $c(k,n)$ as these change-of-basis
coefficients:
\[ x^k = \sum_{n=0}^{k} s(k,n)x^n \]
\[ x^r = \sum_{n=0}^{k} c(k,n)x^n \]

(a) (5 points) Compute \( s(4,k) \) and \( c(4,k) \) for \( k = 1, 2, 3, 4 \), and give simple explicit formulas for \( s(k,k), c(k,k), s(k,1), c(k,1) \).

(b) (5 points) Explain why \( c(k,n) \) is always non-negative, and write down the simple formula relating \( s(k,n) \) to \( c(k,n) \). (For this reason, when proving facts about \( s(k,n) \) or \( c(k,n) \), one has a choice about which one to use, and one or the other may be more convenient).

(c) (5 points) This is essentially Problem #147. Find a recurrence that expresses \( s(k,n) \) in terms \( s(k-1,n) \) and \( s(k-1,n-1) \).

(d) (5 points) A permutation of \([k]\) is a bijection \( \pi : [k] \to [k] \). We will use the following notation:
\[ \pi = \begin{pmatrix}
  1 & 2 & \cdots & n \\
  \pi(1) & \pi(2) & \cdots & \pi(k)
\end{pmatrix} \]
Any permutation decomposes uniquely into cycles. For example,
\[ \pi = \begin{pmatrix}
  1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
  3 & 2 & 9 & 6 & 4 & 5 & 1 & 8 & 7
\end{pmatrix} \]
decomposes into 4 cycles
\[ 1 \to 3 \to 9 \to 7 \to 1 \]
\[ 4 \to 6 \to 5 \to 4 \]
\[ 2 \to 2 \]
\[ 8 \to 8 \]
Show that \( c(k,n) \) is the number of permutations of \([k]\) having exactly \( n \) cycles.

(e) (No points; just an “Extra for experts”, not required for the exam) Explain why for any \( m \), the two matrices
\[ (S(k,n))_{n,k=0,1,\ldots,m}, \quad (s(k,n))_{n,k=0,1,\ldots,m} \]
are inverse to each other.