Math 5707 Graph theory
Spring 2012, Vic Reiner
Final exam - Due Wednesday May 2, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are not allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total) Chapter I, Exercise 60 from Bollobás: Let $G$ be a (simple) planar graph $G$ with $n \geq 3$ vertices, and let us assume $G$ is connected for simplicity.

(a) (10 points) Show the degree sequence $d(G) = (d_1, \ldots, d_n)$ satisfies

$$\sum_{i=1}^{n} (6 - d_i) \geq \sum_{i=1}^{n} (6 - d_i) \geq 12.$$  

(b) (10 points) Explain why this implies that ...

(i) if the min degree $\delta(G) \geq 5$ then $G$ has at least 12 degree 5 vertices,
(ii) and if $\delta(G) \geq 4$ then $G$ has at least 6 vertices of degree 4 or 5.

2. (15 points) Chapter I, Exercise 71 from Bollobás: given $G$ a graph with minimum degree $\delta(G) \geq 2$, show there exists a connected graph $H$ with the same degrees: $d_H(x) = d_G(x)$ for all $x$ in $V = V(H) = V(G)$.

3. (15 points) Chapter IV, Exercise 8 from Bollobás: Show that a tree with $2k$ leaves contains $k$ edge-disjoint paths joining distinct pairs of leaves.

4. (10 points) Chapter V, Exercise 35 from Bollobás: Show that if a graph $G$ has chromatic number $\chi(G) = k$ then one can orient the edges of $G$ in such a way that the longest directed path has $k$ vertices.
5. (20 points total) For a multigraph $G = (V, E)$ and vertex $m$-coloring $f : V \rightarrow \{1, 2, \ldots, m\}$, let $\text{mono}(f)$ be the number of edges $e = \{x, y\}$ in $E$ which are $f$-monochromatic, that is, $f(x) = f(y)$. Define

$$\mu_G(m, u) = \sum_{f : V \rightarrow \{1, 2, \ldots, m\}} u^{\text{mono}(f)}.$$

(a) (10 points) Prove that when an edge $e$ in $E$ is neither a loop nor a bridge in $G$, one has

$$\mu_G(m, u) = u \cdot \mu_{G/e}(m, u) + \mu_{G \setminus e}(m, u) - \mu_{G/e}(m, u).$$

(b) (10 points) Use (a) to prove that

$$\mu_G(m, u) = m^{k(G)}(u - 1)^{r(G)}T_G\left(\frac{u + m - 1}{u - 1}, u\right).$$

where $k(G)$ denotes the number of connected components of $G$.

6. (20 points total) Let $\mathbb{Z}/k\mathbb{Z}$ be the integers $\{0, 1, 2, \ldots, k - 1\}$ with addition modulo $k$, that is, after adding $x + y$, one takes the remainder in the range $[0, k - 1]$ upon division by $k$. Note that 0 is still an additive identity in $\mathbb{Z}/k\mathbb{Z}$, meaning $0 + x = x = 0$, and every element $x$ in $\mathbb{Z}/k\mathbb{Z}$ has an additive inverse, $-x = k - x$ satisfying $x + (-x) = 0$.

Given an undirected multigraph $G = (V, E)$, pick an arbitrary orientation of its edges to make it a digraph $D = (V, A)$, and then say that an edge-labelling $f : A \rightarrow \mathbb{Z}/k\mathbb{Z}$ is a $\mathbb{Z}/k\mathbb{Z}$-valued flow in $G$ if for every $x$ in $V$ one has

$$\sum_{a=(y,x) \in A} f(a) = \sum_{a=(x,y) \in A} f(a).$$

Say that $f$ is a nowhere-zero flow if in addition $f(a) \neq 0$ for all $a$ in $A$. Let $\varphi_G(k)$ denote the number of nowhere-zero $\mathbb{Z}/k\mathbb{Z}$-valued flows in $G$.

(a) (5 points) Explain why $\varphi_G(k)$ does not depend upon the choice of the orientation and digraph $D$ used in the definition, that is, why two different orientations $D, D'$ of $G$ will have the same number of nowhere-zero $\mathbb{Z}/k\mathbb{Z}$-valued flows.

(b) (10 points) Prove that if $e$ is a non-loop edge of $G$ that

$$\varphi_{G/e}(k) = \varphi_G(k) + \varphi_{G \setminus e}(k).$$

(c) (5 points) Prove that $\varphi_G(k) = (-1)^{n(G)}T_G(0, 1 - k)$ where recall that $n(G) = |E| - |V| + k(G)$ is the nullity of $G$. 