Math 5707 Graph theory
Spring 2012, Vic Reiner
Final exam - Due Wednesday May 2, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points total) Chapter I, Exercise 60 from Bollobás: Let G be a (simple) planar graph G with $n \geq 3$ vertices, and let us assume G is connected for simplicity.

(a) (10 points) Show the degree sequence $d(G) = (d_1, \ldots, d_n)$ satisfies

$$\sum_{\substack{i=1,2,\ldots,n \\ \text{with } d_i \leq 6}} (6 - d_i) \geq \sum_{i=1}^{n} (6 - d_i) \geq 12.$$

(b) (10 points) Explain why this implies that ...
(i) if the min degree $\delta(G) \geq 5$ then G has at least 12 degree 5 vertices,
(ii) and if $\delta(G) \geq 4$ then G has at least 6 vertices of degree 4 or 5.

2. (15 points) Chapter I, Exercise 71 from Bollobás: given G a graph with minimum degree $\delta(G) \geq 2$, show there exists a *connected* graph H with the same degrees: $d_H(x) = d_G(x)$ for all x in $V = V(H) = V(G)$.

3. (15 points) Chapter IV, Exercise 8 from Bollobás: Show that a tree with $2k$ leaves contains k edge-disjoint paths joining distinct pairs of leaves.

4. (10 points) Chapter V, Exercise 35 from Bollobás: Show that if a graph G has chromatic number $\chi(G) = k$ then one can orient the edges of G in such a way that the longest directed path has k vertices.

5. (20 points total) For a multigraph $G = (V, E)$ and vertex m-coloring $f : V \to \{1, 2, \ldots, m\}$, let $\mathrm{mono}(f)$ be the number of edges $e = \{x, y\}$ in E which are f-monochromatic, that is, $f(x) = f(y)$. Define

$$\mu_G(m, u) = \sum_{f:V\to\{1,2,\ldots,m\}} u^{\mathrm{mono}(f)}.$$

(a) (10 points) Prove that when an edge e in E is neither a loop nor a bridge in G, one has

$$\mu_G(m, u) = u \cdot \mu_{G/e}(m, u) + \mu_{G\setminus e}(m, u) - \mu_{G/e}(m, u)$$

(b) (10 points) Use (a) to prove that

$$\mu_G(m, u) = m^{k(G)}(u - 1)^{r(G)} T_G\left(\frac{u + m - 1}{u - 1}, u\right).$$

where $k(G)$ denotes the number of connected components of G.

6. (20 points total) Let $\mathbb{Z}/k\mathbb{Z}$ be the integers $\{0, 1, 2, \ldots, k - 1\}$ with addition modulo k, that is, after adding $x + y$, one takes the remainder in the range $[0, k-1]$ upon division by k. Note that 0 is still an additive identity in $\mathbb{Z}/k\mathbb{Z}$, meaning $0 + x = x + 0 = x$, and every element x in $\mathbb{Z}/k\mathbb{Z}$ has an additive inverse, $-x = k - x$ satisfying $x + (-x) = 0$.

Given an undirected multigraph $G = (V, E)$, pick an arbitrary orientation of its edges to make it a digraph $D = (V, A)$, and then say that an edge-labelling $f : A \to \mathbb{Z}/k\mathbb{Z}$ is a $\mathbb{Z}/k\mathbb{Z}$-*valued flow* in G if for every x in V one has

$$\sum_{a=(y,x)\in A} f(a) = \sum_{a=(x,y)\in A} f(a).$$

Say that f is a *nowhere-zero flow* if in addition $f(a) \neq 0$ for all a in A. Let $\varphi_G(k)$ denote the number of nowhere-zero $\mathbb{Z}/k\mathbb{Z}$-valued flows in G.

(a) (5 points) Explain why $\varphi_G(k)$ does not depend upon the choice of the orientation and digraph D used in the definition, that is, why two different orientations D, D' of G will have the same number of nowhere-zero $\mathbb{Z}/k\mathbb{Z}$-valued flows.

(b) (10 points) Prove that if e is a non-loop edge of G that

$$\varphi_{G/e}(k) = \varphi_G(k) + \varphi_{G\setminus e}(k).$$

(c) (5 points) Prove that $\varphi_G(k) = (-1)^{n(G)} T_G(0, 1 - k)$ where recall that $n(G) = |E| - |V| + k(G)$ is the nullity of G.