Math 5707 Graph theory
Spring 2012, Vic Reiner
Midterm exam 1- Due Wednesday Feb. 29, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (10 points) Prove every sequence $(d_1, \ldots, d_n)$ of nonnegative integers d_i for which $\sum_{i=1}^{n} d_i$ is even is the degree sequence for at least one **multigraph** G on n vertices, that is, allowing G to have self-loops and multiple edges.

2. (15 points) Prove part of an assertion from lecture: after reindexing the vertex degrees $d_i = d_G(i)$ of a simple graph G on vertices $V = \{1, 2, \ldots, n\}$ so $d_1 \geq d_2 \geq \cdots \geq d_n$, for every $k = 1, 2, \ldots, n$ one has the inequality

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{d_i, k\}.$$

3. (15 points) Chapter I, Exercise 80 from Bollobás: Show that every connected (simple, undirected) graph $G = (V, E)$ with $m = |E|$ **even** has an orientation of its edges making a digraph $D = (V, A)$ in which every vertex has **even** outdegree.

4. Define a digraph $D(k, n) = (V, A)$ whose vertex set V consists of all words $(a_1, a_2, \ldots, a_n)$ of length n from an alphabet of k letters in which $a_i \neq a_{i+1}$ for $i = 1, 2, \ldots, n-1$, and the arc set A consists of all arcs of this form:

$$(a_0, a_1, \ldots, a_{n-1}) \longrightarrow (a_1, \ldots, a_{n-1}, a_n).$$

(a) (2 points) Draw the digraph $D(3, 2)$.

(b) (3 points) Show that $|V| = k(k-1)^{n-1}$ and $|A| = k(k-1)^n$.

(c) (10 points) Prove for all k, n that $D(k, n)$ has a directed Euler tour.

5. Let $G = (X \sqcup Y, E)$ be a bipartite graph for which there exist positive integers d_X, d_Y such that every x in X has the same degree $d_G(x) = d_X$ and every y in Y has the same degree $d_G(y) = d_Y$.

(a) (5 points) Prove that $d_X/d_Y = |Y|/|X|$.

(b) (10 points) Prove that if $d_X \geq d_Y$ then there exists a matching $M \subseteq E$ that matches every x in X.

6. For an undirected multigraph $G = (V, E)$, an *orientation* ω of G is a choice of one direction for each edge of E, making it a directed arc.

(a) (3 points) Explain why the number of orientations of G is $2^{\hat{m}}$ where $\hat{m}$ denotes the number of edges of E which are not self-loops.

Say that the orientation ω of G is an *acyclic orientation* if it contains no directed cycles; in particular, this requires that G have no self-loops. Let $\mathrm{ac}(G)$ denote the number of acyclic orientations of G.

(b) (2 points) Show the complete graph K_3 has $\mathrm{ac}(K_3) = 6$ by drawing all 6 of its acyclic orientations.

Given an undirected multigraph $G = (V, E)$ and a non-loop edge e, fix some acyclic orientation of the deletion $G \setminus e$, and then consider the two possible orientations of e, some of which may make G acyclic.

Let a_0, a_1, a_2, respectively, denote the number of acyclic orientations of $G \setminus e$ for which $0, 1$, or 2, respectively, out of these possible orientations of e extend it acyclically to all of G.

(c) (5 points) Prove $a_0 = 0$ and $a_1 + a_2 = \mathrm{ac}(G \setminus e)$.

(d) (5 points) Prove $a_1 + 2a_2 = \mathrm{ac}(G)$.

(e) (10 points) Prove $a_2 = \mathrm{ac}(G/e)$, where G/e is the contraction of e in G, and therefore why

$$\mathrm{ac}(G) = \mathrm{ac}(G \setminus e) + \mathrm{ac}(G/e)$$

for any non-loop edge e of G.

(f) (5 points) Use these initial conditions

$$\mathrm{ac}(G) = 0 \text{ if there are any self-loops in } G,$$
$$\mathrm{ac}(G) = 1 \text{ if there are no edges at all in } G.$$

together with the last equation in (e) to illustrate how you can compute $\mathrm{ac}(K_3)$ via recursion on the number of edges.