1. (10 points) Prove every sequence \((d_1, \ldots, d_n)\) of nonnegative integers \(d_i\) for which \(\sum_{i=1}^{n} d_i\) is even is the degree sequence for at least one multigraph \(G\) on \(n\) vertices, that is, allowing \(G\) to have self-loops and multiple edges.

2. (15 points) Prove part of an assertion from lecture: after reindexing the vertex degrees \(d_i = d_G(i)\) of a simple graph \(G\) on vertices \(V = \{1, 2, \ldots, n\}\) so \(d_1 \geq d_2 \geq \cdots \geq d_n\), for every \(k = 1, 2, \ldots, n\) one has the inequality
\[
\sum_{i=1}^{k} d_i \leq k(k - 1) + \sum_{i=k+1}^{n} \min\{d_i, k\}.
\]

3. (15 points) Chapter I, Exercise 80 from Bollobás: Show that every connected (simple, undirected) graph \(G = (V, E)\) with \(m = |E|\) even has an orientation of its edges making a digraph \(D = (V, A)\) in which every vertex has even outdegree.

4. Define a digraph \(D(k, n) = (V, A)\) whose vertex set \(V\) consists of all words \((a_1, a_2, \ldots, a_n)\) of length \(n\) from an alphabet of \(k\) letters in which \(a_i \neq a_{i+1}\) for \(i = 1, 2, \ldots, n - 1\), and the arc set \(A\) consists of all arcs of this form:
\[ (a_0, a_1, \ldots, a_{n-1}) \rightarrow (a_1, \ldots, a_{n-1}, a_n). \]

(a) (2 points) Draw the digraph \(D(3, 2)\).

(b) (3 points) Show that \(|V| = k(k - 1)^{n-1}\) and \(|A| = k(k - 1)^n\).

(c) (10 points) Prove for all \(k, n\) that \(D(k, n)\) has a directed Euler tour.
5. Let \( G = (X \sqcup Y, E) \) be a bipartite graph for which there exist positive integers \( d_X, d_Y \) such that every \( x \) in \( X \) has the same degree \( d_G(x) = d_X \) and every \( y \) in \( Y \) has the same degree \( d_G(y) = d_Y \).

(a) (5 points) Prove that \( d_X/d_Y = |Y|/|X| \).

(b) (10 points) Prove that if \( d_X \geq d_Y \) then there exists a matching \( M \subseteq E \) that matches every \( x \) in \( X \).

6. For an undirected multigraph \( G = (V, E) \), an orientation \( \omega \) of \( G \) is a choice of one direction for each edge of \( E \), making it a directed arc.

(a) (3 points) Explain why the number of orientations of \( G \) is \( 2^\hat{m} \) where \( \hat{m} \) denotes the number of edges of \( E \) which are not self-loops.

Say that the orientation \( \omega \) of \( G \) is an acyclic orientation if it contains no directed cycles; in particular, this requires that \( G \) have no self-loops. Let \( ac(G) \) denote the number of acyclic orientations of \( G \).

(b) (2 points) Show the complete graph \( K_3 \) has \( ac(K_3) = 6 \) by drawing all 6 of its acyclic orientations.

Given an undirected multigraph \( G = (V, E) \) and a non-loop edge \( e \), fix some acyclic orientation of the deletion \( G \setminus e \), and then consider the two possible orientations of \( e \), some of which may make \( G \) acyclic.

Let \( a_0, a_1, a_2 \), respectively, denote the number of acyclic orientations of \( G \setminus e \) for which 0, 1, or 2, respectively, out of these possible orientations of \( e \) extend it acyclically to all of \( G \).

(c) (5 points) Prove \( a_0 = 0 \) and \( a_1 + a_2 = ac(G \setminus e) \).

(d) (5 points) Prove \( a_1 + 2a_2 = ac(G) \).

(e) (10 points) Prove \( a_2 = ac(G/e) \), where \( G/e \) is the contraction of \( e \) in \( G \), and therefore why

\[
ac(G) = ac(G \setminus e) + ac(G/e)
\]

for any non-loop edge \( e \) of \( G \).

(f) (5 points) Use these initial conditions

\[
ac(G) = 0 \text{ if there are any self-loops in } G,
\]
\[
ac(G) = 1 \text{ if there are no edges at all in } G.
\]

together with the last equation in (e) to illustrate how you can compute \( ac(K_3) \) via recursion on the number of edges.