Math 5707 Graph theory
Spring 2012, Vic Reiner
Midterm exam 2- Due Wednesday Apr 11, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (10 points) Chapter III, Exercise 74 from Bollobás: For with k, ℓ having $1 \leq k \leq \ell$, exhibit a simple graph $G_{k,\ell}$ simultaneously having

- vertex-connectivity $\kappa(G_{k,\ell}) = k$, and
- edge-connectivity $\lambda(G_{k,\ell}) = \ell$.

2. (20 points total) Given a simple graph $G = (V, E)$, recall that its complement $\bar{G} = (V, \bar{E})$ has the same vertex set V, but the complementary set of edges $\bar{E} := \{\{x, y\} \subset V : \{x, y\} \notin E\}$. Prove the following inequalities involving the chromatic numbers $\chi(G)$ and $\chi(\bar{G})$.

(a) (5 points) Show $\chi(\bar{G}) \geq \alpha(G)$, where recall that $\alpha(G)$ is the size of the largest independent/stable set of vertices in G.

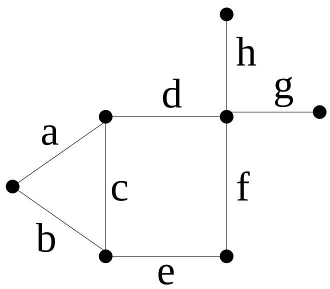
(b) (5 points) Show $\chi(G) \cdot \chi(\bar{G}) \geq n := |V|$.

(c) (10 points) Show $\chi(G) + \chi(\bar{G}) \geq 2\sqrt{n}$.
(Hint: even if you didn't prove (b), you can assume it in part (c).)

3. (15 points) For $n \geq 2$, compute the chromatic polynomial $p_{K_n - \{e\}}(k)$ for the graph $K_n - \{e\}$ obtained from from the complete graph K_n on vertices by deleting any one of its edges e. Factor this polynomial $p_{K_n - \{e\}}(k)$ as completely as possible as a polynomial in k.

4. (10 points) Fix $n, k, t \geq 2$, and show that there exists at least one edge-coloring ω (not necessarily proper) of K_n using t colors with the following property: the number of k-cliques K_k inside K_n whose edges are colored mononchromatically (all of the same color) by ω is less than or equal to $\binom{n}{k} t^{1-\binom{k}{2}}$.

5. (20 points total) Given a simple graph $G = (V, E)$, recall that its *line graph* $\mathrm{line}(G) = (V_{\mathrm{line}(G)}, E_{\mathrm{line}(G)})$ has vertex set $V_{\mathrm{line}(G)} = E$, the edge set of G, and has an edge $\{e, e'\}$ in $E_{\mathrm{line}(G)}$ whenever e, e' were incident at some vertex x in V of G. An example is illustrated below.



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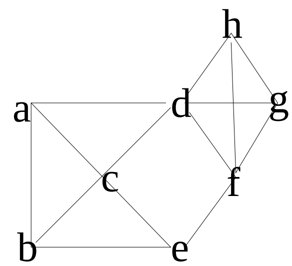

line(G)

(a) (5 points) Show that an edge $e = \{x, y\}$ in E of G gives rise to a vertex of $\mathrm{line}(G)$ having $\deg_{\mathrm{line}(G)}(e) = d_G(x) + d_G(y) - 2$. Explain why this implies both that

(i) a d-regular graph G has $\mathrm{line}(G)$ being $2(d-1)$-regular, and

(ii) a bipartite graph $G = (X \sqcup Y, E)$ which is (d_X, d_Y)-*regular*, in the sense that $d_G(x) = d_X$ and $d_G(y) = d_Y$ for all $x \in X, y \in Y$, has $\mathrm{line}(G)$ being $(d_X + d_Y - 2)$-regular.

(b) (15 points) Prove that a connected simple graph G has $\mathrm{line}(G)$ a k-regular graph for some $k \geq 0$ if and only if either

(i) G is d-regular, with $2d = k + 2$, or

(ii) G is bipartite and (d_X, d_Y)-regular, with d_X, d_y some positive integers whose sum is $k + 2$.

6. (25 points total) For an undirected multigraph $G = (V, E)$, let us change (from Exam 1 and lecture) our definition of an *orientation* ω of G to mean a choice of one of two directions for each edge of E, making it a directed arc, **even for the loop edges**. Thus G has 2^m orientations where $m = |E|$, regardless of whether G contains loop edges.

Say that the orientation ω of G is *totally cyclic* if every directed arc lies in at least one directed cycle. One can show that this is equivalent to the condition that the orientation on each connected component of G is *strongly connected*: for every pair x, y in V in the same connected component of G, there existed directed paths both x to y and y to x.

Let $\mathrm{tc}(G)$ denote the number of totally cyclic orientations of G. For example, the cycle C_n for $n \geq 1$ has $\mathrm{tc}(C_n) = 2$. Our goal here is a deletion-contraction recurrence to compute this quantity $\mathrm{tc}(G)$.

Given an undirected multigraph $G = (V, E)$ and a non-bridge edge e, fix some totally cyclic orientation ω of the contraction G/e, and then consider the two possible orientations of e one could use to extend ω to an orientation of G, some of which may make G totally cyclic. We adopt here the convention for contracting on a loop edge e which says that G/e is the same as the deletion $G \setminus e$ if e is a loop.

Let t_0, t_1, t_2, respectively, denote the number of totally cyclic orientations ω of G/e for which $0, 1$, or 2, respectively, out of these possible orientations of e extend it totally cyclically to all of G.

(a) (5 points) Prove $t_0 = 0$ and $t_1 + t_2 = \mathrm{tc}(G/e)$.

(b) (5 points) Prove $t_1 + 2t_2 = \mathrm{tc}(G)$.

(c) (10 points) Prove $t_2 = \mathrm{tc}(G \setminus e)$, where $G \setminus e$ is the deletion of e in G, and therefore why

$$\mathrm{tc}(G) = \mathrm{tc}(G \setminus e) + \mathrm{tc}(G/e) \tag{1}$$

for any non-bridge edge e of G.

(d) (5 points) Explain why

$$\mathrm{tc}(G) = 0 \text{ if there are any bridges in } G,$$
$$\mathrm{tc}(G) = 1 \text{ if there are no edges at all in } G.$$

and show how one can use these together with equation (1) to compute $\mathrm{tc}(C_n)$ via recursion on the number of edges.