Math 5707 Graph theory
Spring 2012, Vic Reiner
Midterm exam 2- Due Wednesday Apr 11, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are not allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (10 points) Chapter III, Exercise 74 from Bollobás: For with $k, \ell$ having $1 \leq k \leq \ell$, exhibit a simple graph $G_{k, \ell}$ simultaneously having
   - vertex-connectivity $\kappa(G_{k, \ell}) = k$, and
   - edge-connectivity $\lambda(G_{k, \ell}) = \ell$.

2. (20 points total) Given a simple graph $G = (V, E)$, recall that its complement $\bar{G} = (V, \bar{E})$ has the same vertex set $V$, but the complementary set of edges $\bar{E} := \{(x, y) \subset V : \{x, y\} \notin E\}$. Prove the following inequalities involving the chromatic numbers $\chi(G)$ and $\chi(\bar{G})$.
   (a) (5 points) Show $\chi(\bar{G}) \geq \alpha(G)$, where recall that $\alpha(G)$ is the size of the largest independent/stable set of vertices in $G$.
   (b) (5 points) Show $\chi(G) \cdot \chi(\bar{G}) \geq n := |V|$.
   (c) (10 points) Show $\chi(G) + \chi(\bar{G}) \geq 2\sqrt{n}$.
   (Hint: even if you didn’t prove (b), you can assume it in part (c).)

3. (15 points) For $n \geq 2$, compute the chromatic polynomial $p_{K_n - \{e\}}(k)$ for the graph $K_n - \{e\}$ obtained from the complete graph $K_n$ on vertices by deleting any one of its edges $e$. Factor this polynomial $p_{K_n - \{e\}}(k)$ as completely as possible as a polynomial in $k$.

4. (10 points) Fix $n, k, t \geq 2$, and show that there exists at least one edge-coloring $\omega$ (not necessarily proper) of $K_n$ using $t$ colors with the following property: the number of $k$-cliques $K_k$ inside $K_n$ whose edges are colored monochromatically (all of the same color) by $\omega$ is less than or equal to $\binom{n}{k} t^{1 - \binom{k}{2}}$. 
5. (20 points total) Given a simple graph $G = (V, E)$, recall that its line graph $\text{line}(G) = (V_{\text{line}(G)}, E_{\text{line}(G)})$ has vertex set $V_{\text{line}(G)} = E$, the edge set of $G$, and has an edge $\{e, e'\}$ in $E_{\text{line}(G)}$ whenever $e, e'$ were incident at some vertex $x$ in $V$ of $G$. An example is illustrated below.

(a) (5 points) Show that an edge $e = \{x, y\}$ in $E$ of $G$ gives rise to a vertex of $\text{line}(G)$ having $\deg_{\text{line}(G)}(e) = d_G(x) + d_G(y) - 2$. Explain why this implies both that

(i) a $d$-regular graph $G$ has $\text{line}(G)$ being $2(d - 1)$-regular, and
(ii) a bipartite graph $G = (X \sqcup Y, E)$ which is $(d_X, d_Y)$-regular, in the sense that $d_G(x) = d_X$ and $d_G(y) = d_Y$ for all $x \in X, y \in Y$, has $\text{line}(G)$ being $(d_X + d_Y - 2)$-regular.

(b) (15 points) Prove that a connected simple graph $G$ has $\text{line}(G)$ a $k$-regular graph for some $k \geq 0$ if and only if either

(i) $G$ is $d$-regular, with $2d = k + 2$, or
(ii) $G$ is bipartite and $(d_X, d_Y)$-regular, with $d_X, d_Y$ some positive integers whose sum is $k + 2$. 

6. (25 points total) For an undirected multigraph $G = (V, E)$, let us change (from Exam 1 and lecture) our definition of an orientation $\omega$ of $G$ to mean a choice of one of two directions for each edge of $E$, making it a directed arc, even for the loop edges. Thus $G$ has $2^m$ orientations where $m = |E|$, regardless of whether $G$ contains loop edges.

Say that the orientation $\omega$ of $G$ is totally cyclic if every directed arc lies in at least one directed cycle. One can show that this is equivalent to the condition that the orientation on each connected component of $G$ is strongly connected: for every pair $x, y$ in $V$ in the same connected component of $G$, there existed directed paths both $x$ to $y$ and $y$ to $x$.

Let $tc(G)$ denote the number of totally cyclic orientations of $G$. For example, the cycle $C_n$ for $n \geq 1$ has $tc(C_n) = 2$. Our goal here is a deletion-contraction recurrence to compute this quantity $tc(G)$.

Given an undirected multigraph $G = (V, E)$ and a non-bridge edge $e$, fix some totally cyclic orientation $\omega$ of the contraction $G/e$, and then consider the two possible orientations of $e$ one could use to extend $\omega$ to an orientation of $G$, some of which may make $G$ totally cyclic. We adopt here the convention for contracting on a loop edge $e$ which says that $G/e$ is the same as the deletion $G \setminus e$ if $e$ is a loop.

Let $t_0, t_1, t_2$, respectively, denote the number of totally cyclic orientations $\omega$ of $G/e$ for which 0, 1, or 2, respectively, out of these possible orientations of $e$ extend it totally cyclically to all of $G$.

(a) (5 points) Prove $t_0 = 0$ and $t_1 + t_2 = tc(G/e)$.

(b) (5 points) Prove $t_1 + 2t_2 = tc(G)$.

(c) (10 points) Prove $t_2 = tc(G \setminus e)$, where $G \setminus e$ is the deletion of $e$ in $G$, and therefore why

\[ tc(G) = tc(G \setminus e) + tc(G/e) \]

for any non-bridge edge $e$ of $G$.

(d) (5 points) Explain why

\[ tc(G) = 0 \] if there are any bridges in $G$,
\[ tc(G) = 1 \] if there are no edges at all in $G$.

and show how one can use these together with equation (1) to compute $tc(C_n)$ via recursion on the number of edges.