Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are not allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points) Schrijver’s Problem 2.25 on p. 37. He asserts the equality (formula (64)) of a max and a min, but forgot to ask you to prove it. Prove it.

2. (20 points) Schrijver Problem 1.3 on p. 18. Draw the associated directed graph, along with the last Bellman-Ford function on its vertices, and a tree rooted at vertex $A$ consisting of shortest $(A,v)$-paths for all other vertices $v$.

3. (20 points) In Schrijver’s knapsack problem example (Application 1.3), assume that object 2 is no longer available, so one has

<table>
<thead>
<tr>
<th>article</th>
<th>volume</th>
<th>value</th>
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<tbody>
<tr>
<td>1</td>
<td>5</td>
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<td>3</td>
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<tr>
<td>5</td>
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</tbody>
</table>

Solve this new knapsack problem by drawing an appropriate directed graph and using the Bellman-Ford algorithm.
4. (20 points) Give an example of a directed graph \( D = (V, A) \) and a
length function on the arcs \( \ell : A \to \mathbb{Z} \) with these properties:

- \( |V| = 3 \), i.e. there are exactly 3 vertices, labelled \( V = \{s, r, t\} \),
- there are no directed cycles at all (and therefore none of negative
  length), and
- Dijkstra’s algorithm fails to find an \( s \rightarrow t \) directed path of mini-
  mum length, but the Bellman-Ford algorithm works. (Write
down the output from both algorithms)

5. Let \( G = (V, E) \) be the graph which is a four-vertex cycle on vertices
\( V = a, b, c, d \) with edges and lengths as indicated below:

\[
E = \{ \quad ab, \quad bc, \quad cd, \quad ad \quad \}
\]

\[
\text{length} \quad \begin{array}{cccc}
1 & 3 & 3 & 2
\end{array}
\]

(a) (5 points) List a sequence of forests found during the Dijkstra-Prim
method for computing a minimum spanning tree in \( G \).
(b) (5 points) List a sequence of forests found during Kruskal’s method
for computing a minimum spanning tree in \( G \).
(c) (3 points) Write down all of the minimum spanning trees in \( G \).
(d) (7 points) Prove that for any connected graph \( G \), if all of the edge
lengths \( \ell(e) \) are distinct real numbers then the minimum length span-
ning tree for \( G \) is unique.

(One possible hint for (d): Suppose
\[
T = \{e_1, e_2, \ldots, e_{n-1}\}
\]
\[
T' = \{e'_1, e'_2, \ldots, e'_{n-1}\}
\]
were two minimum length spanning trees in \( G \), with their edges indexed
in increasing order of length, i.e.
\[
\ell(e_1) < \ell(e_2) < \ldots < \ell(e_{n-1})
\]
\[
\ell(e'_1) < \ell(e'_2) < \ldots < \ell(e'_{n-1}),
\]
and suppose that \( e_1 = e'_1, e_2 = e'_2, \ldots, e_k = e'_k \), but \( \ell(e_{k+1}) < \ell(e'_{k+1}) \).
Can you reach a contradiction from this?)